2. The Fibonacci Search

Bracketing Search Methods

An approach for finding the minimum of $f(x)$ in a given interval is to evaluate the function many times and search for a local minimum. To reduce the number of function evaluations it is important to have a good strategy for determining where $f(x)$ is to be evaluated. Two efficient bracketing methods are the golden ratio and Fibonacci searches. To use either bracketing method for finding the minimum of $f(x)$, a special condition must be met to ensure that there is a proper minimum in the given interval.

Definition (Unimodal Function) The function $f(x)$ is unimodal on $I = [a, b]$, if there exists a unique number $p \in I$ such that

$f(x)$ is decreasing on $[a, p]$,
and
$f(x)$ is increasing on $[p, b]$.

Fibonacci Search

In the golden ratio search two function evaluations are made at the first iteration and then only one function evaluation is made for each subsequent iteration. The value of r remains constant on each subinterval and the search is terminated at the k^{th} subinterval, provided that $|b_k - a_k| < \delta$ or $|f(b_k) - f(a_k)| < \epsilon$ where δ, ϵ are the predefined tolerances. The Fibonacci search method differs from the golden ratio method in that the value of r is not constant on each subinterval. Additionally, the number of subintervals (iterations) is predetermined and based on the specified tolerances.

The Fibonacci search is based on the sequence of Fibonacci numbers which are defined by the equations

$F_0 = 0$

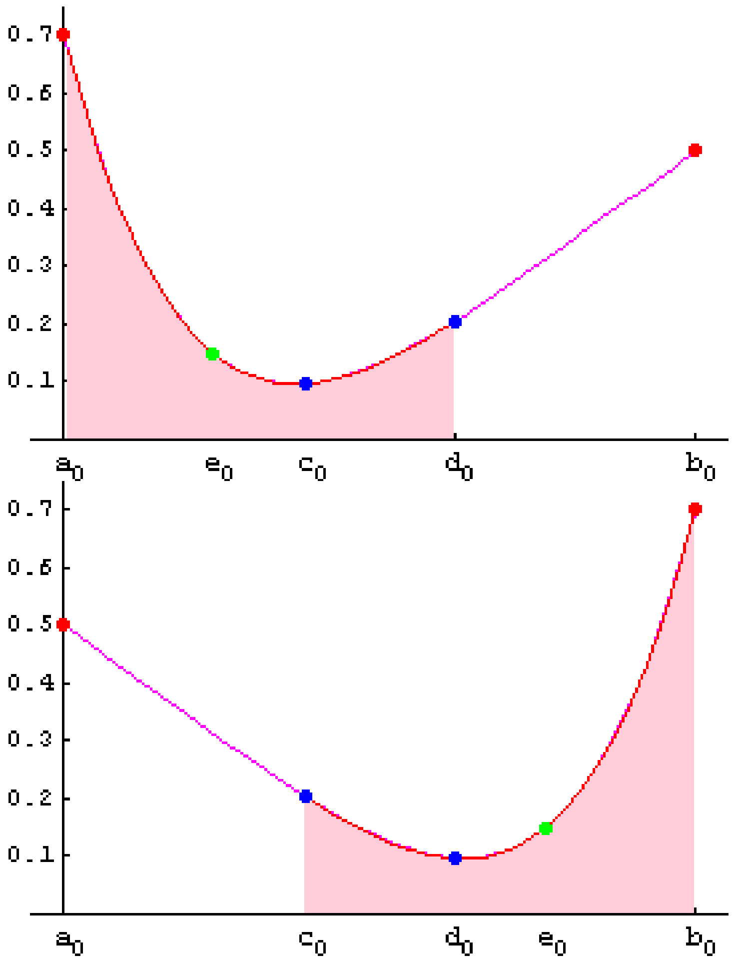
$F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$ for $n = 2, 3, \ldots$

Thus the Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

Assume we are given a function $f(x)$ that is unimodal on the interval $[a_0, b_0]$. As in the golden ratio search a value r_0 $\left(\frac{1}{2} < r_0 < 1\right)$ is selected so that both of the interior points c_0 and d_0 will be used in the next subinterval and there will be only one new function evaluation.

If $f(c_0) \le f(d_0)$, then the minimum must occur in the subinterval $[a_0, d_0]$, and we replace $a_1 = a_0$ and $b_1 = d_0$ and continue the search in the new subinterval $[a_1, b_1] = [a_0, d_0]$. If $f(c_0) > f(d_0)$, then the minimum must occur in the subinterval $[c_0, b_0]$, and we replace $a_1 = c_0$ and $b_1 = b_0$ and continue the search in the new subinterval $[a_1, b_1] = [c_0, b_0]$. These choices are shown in Figure 1 below.



If $f(c_0) \le f(d_0)$, then squeeze from the right and left and use the new interval $[a_1, b_1] = [a_0, d_0]$.

If $f(c_0) < f(d_0)$, then squeeze from the use the new interval $[a_1, b_1] = [c_0, b_0]$.

Figure 1. The decision process for the Fibonacci ratio search.

If $f(c_0) \le f(d_0)$ and only one new function evaluation is to be made in the interval $[a_0, d_0]$, then we select r_1 $\left(\frac{1}{2} < r_1 < 1\right)$ for the subinterval $[a_1, b_1] = [a_0, d_0]$. We already have relabeled

$$b_1 = d_0$$

and since $c_0 \in [a_0, d_0]$ we will relabel it by

$$d_1 = c_0$$

then we will have

(1) $$d_0 - c_0 = b_1 - d_1 .$$

The ratio r_0 is chosen so that $d_0 - a_0 = r_0 (b_0 - a_0)$ and $c_0 - a_0 = (1 - r_0)(b_0 - a_0)$ and subtraction produces

$$d_0 - c_0 = (d_0 - a_0) - (c_0 - a_0)$$

$$d_0 - c_0 = r_0 (b_0 - a_0) - (1 - r_0)(b_0 - a_0)$$

(2) $$d_0 - c_0 = (2 r_0 - 1)(b_0 - a_0)$$

And the ratio r_1 is chosen so that

(3) $$b_1 - d_1 = (1 - r_1)(b_1 - a_1) .$$

Now substitute (2) and (3) into (1) and get

(4) $$(2 r_0 - 1)(b_0 - a_0) = (1 - r_1)(b_1 - a_1) .$$

Also the length of the interval $[a_1, b_1]$ has been shrunk by the factor r_0. Thus $(b_1 - a_1) = r_0 (b_0 - a_0)$ and using this in (4) produces

(5) $$(2 r_0 - 1)(b_0 - a_0) = (1 - r_1)(r_0 (b_0 - a_0)) .$$

Cancel the common factor $(b_0 - a_0)$ in (5) and we now have

(6) $$(2 r_0 - 1) = (1 - r_1) r_0 .$$

Solving (6) for r_1 produces

(7) $$r_1 = \frac{1 - r_0}{r_0} .$$

Now we introduce the Fibonacci numbers $\{F_n = F_{n-1} + F_{n-2}\}$ for the subscript $n \geq 4$. In equation (7), substitute $r_0 = \frac{F_{n-1}}{F_n}$ and get the following

$$r_1 = \left(1 - \frac{F_{n-1}}{F_n}\right) \Big/ \left(\frac{F_{n-1}}{F_n}\right)$$

$$r_1 = \frac{F_n - F_{n-1}}{F_{n-1}}$$

$$r_1 = \frac{F_{n-2}}{F_{n-1}}$$

Reasoning inductively, it follows that the Fibonacci search can be begun with

$$r_0 = \frac{F_{n-1}}{F_n}$$

$$r_1 = \frac{F_{n-2}}{F_{n-1}}$$

and

$$r_k = \frac{F_{n-1-k}}{F_{n-k}} \quad \text{for } k = 1, 2, \ldots, n-3.$$

Note that the last step will be

$$r_{n-3} = \frac{F_2}{F_3} = \frac{1}{2},$$

thus no new points can be added at this stage (i.e. the algorithm terminates). Therefore, the set of possible ratios is

$$\{r_k\}_{k=0}^{k=n-3}.$$

There will be exactly $n-2$ steps in a Fibonacci search!

The $(k+1)^{st}$ subinterval is obtained by reducing the length of the k^{th} subinterval by a factor of $r_k = \frac{F_{n-1-k}}{F_{n-k}}$. After $n-2$ steps the length of the last subinterval will be

$$\frac{F_{n-1}}{F_n} \frac{F_{n-2}}{F_{n-1}} \frac{F_{n-3}}{F_{n-2}} \cdots \frac{F_3}{F_4} \frac{F_2}{F_3} (b_0 - a_0) = \frac{F_2}{F_n} (b_0 - a_0) = \frac{1}{F_n} (b_0 - a_0).$$

If the abscissa of the minimum is to be found with a tolerance of ϵ, then we want

$\frac{1}{F_n}(b_0 - a_0) < \epsilon$. It is necessary to use n iterations, where n is the smallest integer such that

(8) $$F_n > \frac{b_0 - a_0}{\epsilon}.$$

Note. Solving the above inequality requires either a trial and error look at the sequence of Fibonacci numbers, or the deeper fact that the Fibonacci numbers can be generated by the formula

$$F_k = \left(\left(1 + \sqrt{5}\right)^k - \left(1 - \sqrt{5}\right)^k\right) / \left(2^k \sqrt{5}\right).$$

Knowing this fact may be useful, but we still need to compute all the Fibonacci numbers $F_0, F_1, F_2, \ldots, F_{n-1}, F_n$ in order to calculate the ratios $\left\{r_k = \frac{F_{n-1-k}}{F_{n-k}}\right\}_{k=0}^{k=n-3}$.

The interior points c_k and d_k of the k^{th} subinterval $[a_k, b_k]$ are found, as needed, using the formulas

(9) $$c_k = a_k + \left(1 - \frac{F_{n-1-k}}{F_{n-k}}\right)(b_k - a_k),$$

(10) $$d_k = a_k + \frac{F_{n-1-k}}{F_{n-k}}(b_k - a_k).$$

Each iteration requires the determination of two new interior points, one from the previous iteration and the second from formula (9) or (10). When $r_{n-3} = \frac{F_2}{F_3} = \frac{1}{2}$, the two interior points will be concurrent in the middle of the interval. In following example, to distinguish the last two interior points a small distinguishability constant, e, is introduced. Thus when $k = n - 3$ is used in formula (9) or (10), the coefficients of $(b_k - a_k)$ are $\frac{1}{2} - e$ or $\frac{1}{2} + e$, respectively.

Example 1. Find the minimum of the unimodal function $f(x) = x^2 - \sin(x)$ on the interval $[0, 1]$ using the Fibonacci search method. Use the tolerance of $\epsilon = 10^{-4}$ and the distinguishability constant $e = 0.01$

Solution 1.

Example 2. Find the minimum of $f[x] = \frac{1}{2} + x^5 - \frac{4}{5}x$ on the interval $[0, 1]$.

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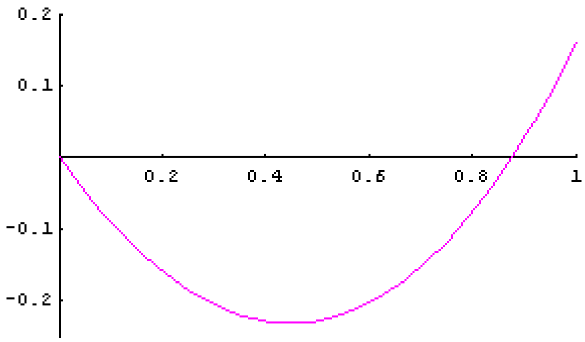
Solution 1.



$$y = f[x] = x^2 - \text{Sin}[x]$$

The smallest Fibonacci number satisfying

$$F_n > \frac{b_0 - a_0}{\epsilon} = \frac{1 - 0}{10^{-4}} = 10000$$

By trial and error we find that $F_{20} = 6765$ and $F_{21} = 10946$.

$$F_{20} = 6765$$
$$F_{21} = 10946$$

Thus we must choose $n = 21$, and the first ratio we must use is

$$r_0 = \frac{F_{20}}{F_{21}} = \frac{6765}{10946} = 0.618034$$

Let $a_0 = 0$ and $b_0 = 1$, and start with the initial interval $[a_0, b_0] = [0, 1]$. Formulas (9) and (10) are used to compute d_0 and d_0 as follows:

$$c_0 = a_0 + (1.0 - \frac{F_{20}}{F_{21}})(b_0 - a_0)$$
$$c_0 = 0 + (1. - 0.618034)(1)$$
$$c_0 = 0 + (0.381966)(1)$$
$$c_0 = 0.381966$$

$$d_0 = a_0 + \frac{F_{20}}{F_{21}}(b_k - a_k)$$
$$d_0 = 0 + (0.618034)(1)$$
$$d_0 = 0.618034$$

$$f[c_0] = f[0.381966] = -0.226847$$
$$f[d_0] = f[0.618034] = -0.197468$$

Thus, the minimum of $f[x]$ will occur in the subinterval $[a_0, d_0]$. We set $a_1 = a_0$, $b_1 = d_0$, and $d_1 = c_0$. The new subinterval containing the abscissa of the minimum of $f[x]$ is $[a_1, b_1] = [a_0, d_0] = [0, 0.618034]$. Now use formulas (9) to calculate the interior point c_1 as follows:

The new subinterval is $[a_1, b_1] = [0, 0.618034]$

$$c_1 = a_1 + \left(1.0 - \frac{F_{19}}{F_{20}}\right)(b_1 - a_1)$$

$$c_1 = 0 + (1. - 0.618034)(0.618034)$$

$$c_1 = 0 + (0.381966)(0.618034)$$

$$c_1 = 0.236068$$

$$f[c_1] = f[0.236068] = -0.178153$$

$$f[d_1] = f[0.381966] = -0.226847$$

Now compute and compare $f[c_1]$ and $f[d_1]$ to determine the new subinterval $[a_2, b_2] = [c_1, b_1] = [0.236068, 0.618034]$, and continue the iteration process. The iterations are obtained by calling the subroutine.

```
f[{ 0.0000000,  0.3819660,  0.6180340,  1.0000000}] = { 0.0000000, -0.2268475, -0.1974679,  0.1585290}
f[{ 0.0000000,  0.2360680,  0.3819660,  0.6180340}] = { 0.0000000, -0.1781534, -0.2268475, -0.1974679}
f[{ 0.2360680,  0.3819660,  0.4721359,  0.6180340}] = {-0.1781534, -0.2268475, -0.2318772, -0.1974679}
f[{ 0.3819660,  0.4721359,  0.5278641,  0.6180340}] = {-0.2268475, -0.2318772, -0.2250488, -0.1974679}
f[{ 0.3819660,  0.4376941,  0.4721359,  0.5278641}] = {-0.2268475, -0.2322759, -0.2318772, -0.2250488}
f[{ 0.3819660,  0.4164078,  0.4376941,  0.4721359}] = {-0.2268475, -0.2310824, -0.2322759, -0.2318772}
f[{ 0.4164078,  0.4376941,  0.4508496,  0.4721359}] = {-0.2310824, -0.2322759, -0.2324650, -0.2318772}
f[{ 0.4376941,  0.4508496,  0.4589804,  0.4721359}] = {-0.2322759, -0.2324650, -0.2323713, -0.2318772}
f[{ 0.4376941,  0.4458250,  0.4508496,  0.4589804}] = {-0.2322759, -0.2324425, -0.2324650, -0.2323713}
f[{ 0.4458250,  0.4508496,  0.4539558,  0.4589804}] = {-0.2324425, -0.2324650, -0.2324482, -0.2323713}
f[{ 0.4458250,  0.4489311,  0.4508496,  0.4539558}] = {-0.2324425, -0.2324637, -0.2324650, -0.2324482}
f[{ 0.4489311,  0.4508496,  0.4520373,  0.4539558}] = {-0.2324637, -0.2324650, -0.2324614, -0.2324482}
f[{ 0.4489311,  0.4501188,  0.4508496,  0.4520373}] = {-0.2324637, -0.2324656, -0.2324650, -0.2324614}
f[{ 0.4489311,  0.4496620,  0.4501188,  0.4508496}] = {-0.2324637, -0.2324652, -0.2324656, -0.2324650}
f[{ 0.4496620,  0.4501188,  0.4503928,  0.4508496}] = {-0.2324652, -0.2324656, -0.2324655, -0.2324650}
f[{ 0.4496620,  0.4499360,  0.4501188,  0.4503928}] = {-0.2324652, -0.2324655, -0.2324656, -0.2324655}
f[{ 0.4499360,  0.4501188,  0.4502101,  0.4503928}] = {-0.2324655, -0.2324656, -0.2324656, -0.2324655}
f[{ 0.4501188,  0.4502101,  0.4503015,  0.4503928}] = {-0.2324656, -0.2324656, -0.2324656, -0.2324655}
f[{ 0.4501188,  0.4502101,  0.4502101,  0.4503015}] = {-0.2324656, -0.2324656, -0.2324656, -0.2324656}
```

```
After k = 19 iterations
{  a,   b,   c,   d}  = { 0.450118765,  0.450210122,  0.450210122,  0.450301480}
{f[a],f[b],f[c],f[d]} = {-0.232465570038, -0.232465574302, -0.232465574302, -0.232465558242}
```

$f[x] = x^2 - \mathrm{Sin}[x]$

In the subroutine, we have used $\frac{F_2}{F_3} = \frac{1}{2}$, and hence $c = d$. If this is acceptable then we have found an approximation to the minimum.

$f[x] = x^2 - \mathrm{Sin}[x]$

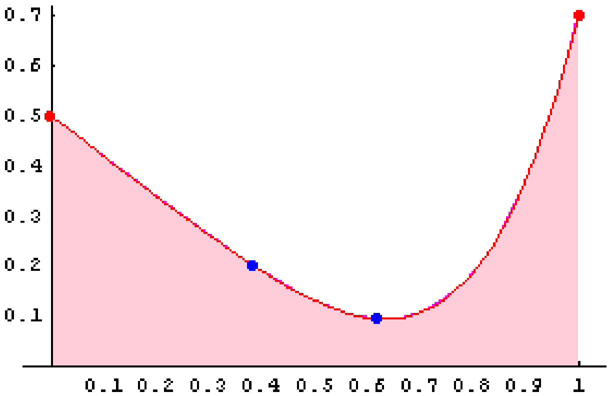
```
An approximation for the minimum is
  c  =    0.45021012
f[c] = f[ 0.45021012]
f[c] = -0.232465574
```

If we use the distinguishability constant $e = 0.01$ in the final computation, then a plausible computation would be

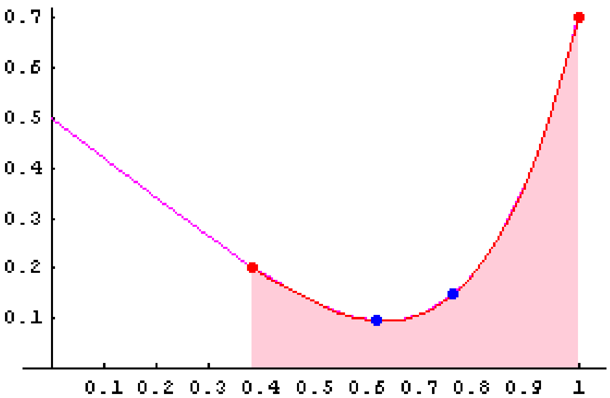
```
{  a,   b,   c,   d}  = { 0.45011876,  0.45020830,  0.45021012,  0.45030148}
{f[a],f[b],f[c],f[d]} = {-0.232465570, -0.232465558, -0.232465574, -0.232465574}
```

Example 2. Find the minimum of $f[x] = \frac{1}{2} + x^5 - \frac{4}{5} x$ on the interval $[0, 1]$.

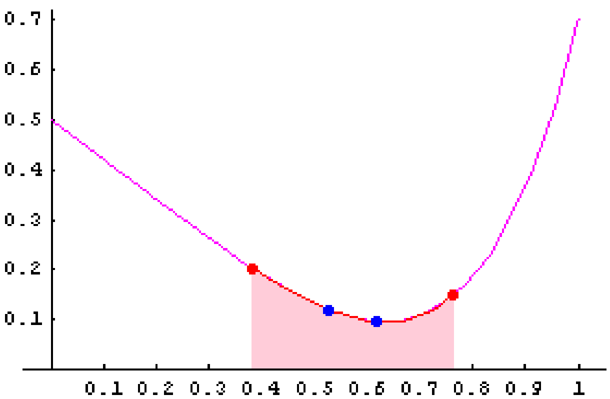
Solution 2.



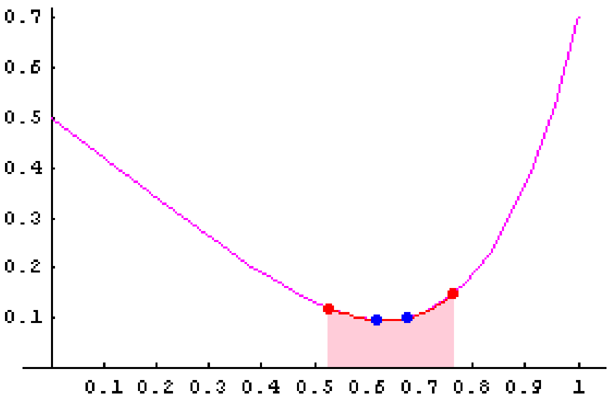
{ a, b, c, d} = { 0.000000, 0.381944, 0.618056, 1.000000}
{f[a],f[b],f[c],f[d]} = { 0.500000000000, 0.202572768079, 0.095741233176, 0.700000000000}



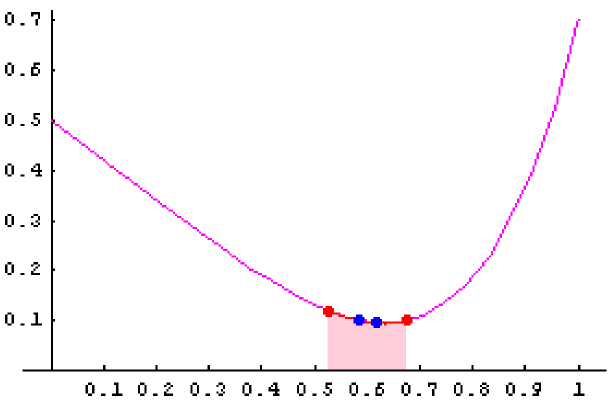
{ a, b, c, d} = { 0.381944, 0.618056, 0.763889, 1.000000}
{f[a],f[b],f[c],f[d]} = { 0.202572768079, 0.095741233176, 0.148995245189, 0.700000000000}



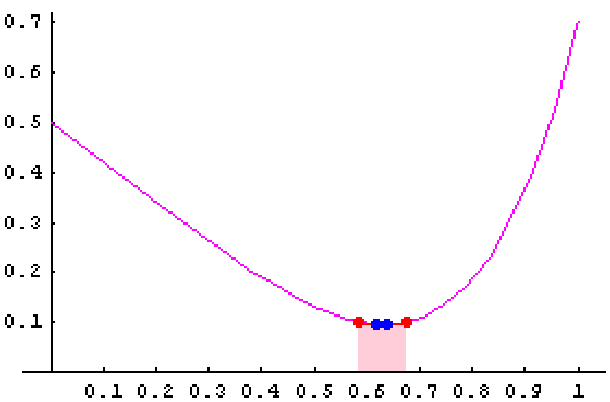
{ a, b, c, d} = { 0.381944, 0.527778, 0.618056, 0.763889}
{f[a],f[b],f[c],f[d]} = { 0.202572768079, 0.118727928156, 0.095741233176, 0.148995245189}



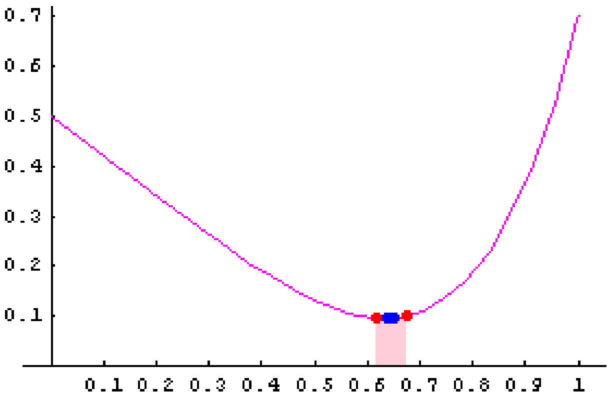
{ a, b, c, d} = { 0.527778, 0.618056, 0.673611, 0.763889}
{f[a],f[b],f[c],f[d]} = { 0.118727928156, 0.095741233176, 0.099801450479, 0.148995245189}



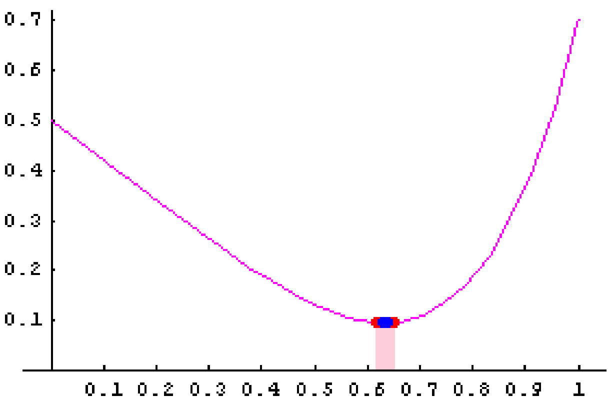
{ a, b, c, d} = { 0.527778, 0.583333, 0.618056, 0.673611}
{f[a],f[b],f[c],f[d]} = { 0.118727928156, 0.100876896862, 0.095741233176, 0.099801450479}



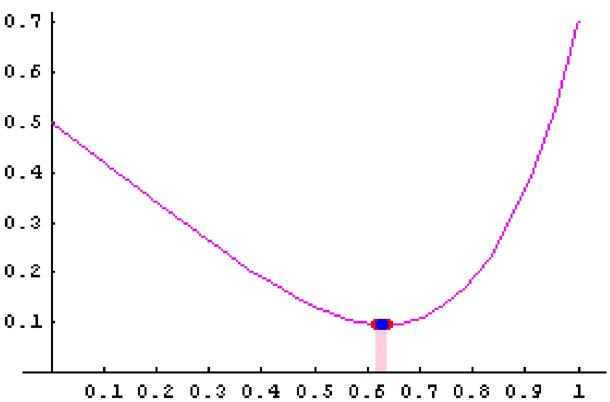
{ a, b, c, d} = { 0.583333, 0.618056, 0.638889, 0.673611}
{f[a],f[b],f[c],f[d]} = { 0.100876896862, 0.095741233176, 0.095334234465, 0.099801450479}



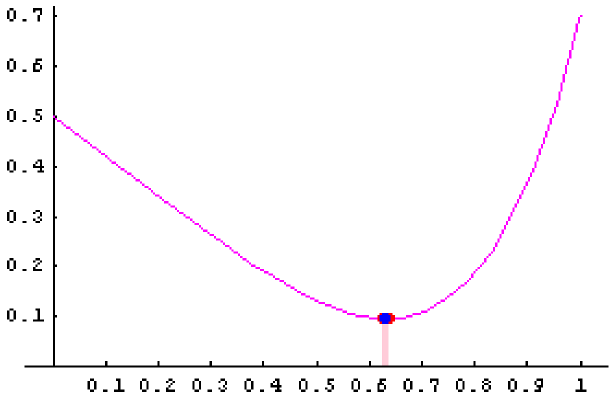
```
{   a,    b,    c,    d}   = { 0.618056,  0.638889,  0.652778,  0.673611}
{f[a],f[b],f[c],f[d]} = { 0.095741233176,  0.095334234465,  0.096307374701,  0.099801450479}
```



```
{   a,    b,    c,    d}   = { 0.618056,  0.631944,  0.638889,  0.652778}
{f[a],f[b],f[c],f[d]} = { 0.095741233176,  0.095229119781,  0.095334234465,  0.096307374701}
```



```
{   a,    b,    c,    d}   = { 0.618056,  0.625000,  0.631944,  0.638889}
{f[a],f[b],f[c],f[d]} = { 0.095741233176,  0.095367431641,  0.095229119781,  0.095334234465}
```



{ a, b, c, d} = { 0.625000, 0.631944, 0.631944, 0.638889}

{f[a],f[b],f[c],f[d]} = { 0.095367431641, 0.095229119781, 0.095229119781, 0.095334234465}

After k = 10 iterations

{ a, b, c, d} = { 0.625000000, 0.631944444, 0.631944444, 0.638888889}

{f[a],f[b],f[c],f[d]} = { 0.095367431641, 0.095229119781, 0.095229119781, 0.095334234465}

$f[x] = \frac{1}{2} - \frac{4x}{5} + x^5$