5. Simpson's 3/8 Rule for Numerical Integration

The numerical integration technique known as "Simpson's 3/8 rule" is credited to the mathematician Thomas Simpson (1710-1761) of Leicestershire, England. His also worked in the areas of numerical interpolation and probability theory.

Theorem (Simpson's 3/8 Rule) Consider $y = f(x)$ over $[x_0, x_3]$, where $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, and $x_3 = x_0 + 3h$. Simpson's 3/8 rule is

$$SE(f, h) = \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)).$$

This is an numerical approximation to the integral of $f(x)$ over $[x_0, x_3]$ and we have the expression

$$\int_{x_0}^{x_3} f(x)\,dx \approx SE(f, h).$$

The remainder term for Simpson's 3/8 rule is $R_{SE}(f, h) = -\frac{3}{80} f^{(4)}(c)\,h^5$, where c lies somewhere between x_0 and x_3, and have the equality

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)) - \frac{3}{80} f^{(4)}(c)\,h^5.$$

Composite Simpson's 3/8 Rule

Our next method of finding the area under a curve $y = f(x)$ is by approximating that curve with a series of cubic segments that lie above the intervals $\{[x_{k-1}, x_k]\}_{k=1}^{3m}$. When several cubics are used, we call it the composite Simpson's 3/8 rule.

Theorem (Composite Simpson's 3/8 Rule) Consider $y = f(x)$ over $[a, b]$. Suppose that the interval $[a, b]$ is subdivided into $3m$ subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{3m}$ of equal width $h = \frac{b-a}{3m}$ by using the equally spaced sample points $x_k = x_0 + kh$ for $k = 0, 1, 2, ..., 3m$. The composite Simpson's 3/8 rule for $3m$ subintervals is

$$SC(f, h) = \frac{3h}{8}\sum_{k=1}^{m}(f(x_{3k-3}) + 3f(x_{3k-2}) + 3f(x_{3k-1}) + f(x_{3k})).$$

is an numerical approximation to the integral, and

$$\int_a^b f(x)\,dx = SC(f, h) + E_{SC}(f, h).$$

Furthermore, if $f(x) \in C^4[a, b]$, then there exists a value c with $a < c < b$ so that the error term $E_{3c}(f, h)$ has the form

$$E_{3c}(f, h) = -\frac{(b - a) f^4(c)}{80} h^4.$$

This is expressed using the "big O" notation $E_{3E}(f, h) = O(h^4)$.

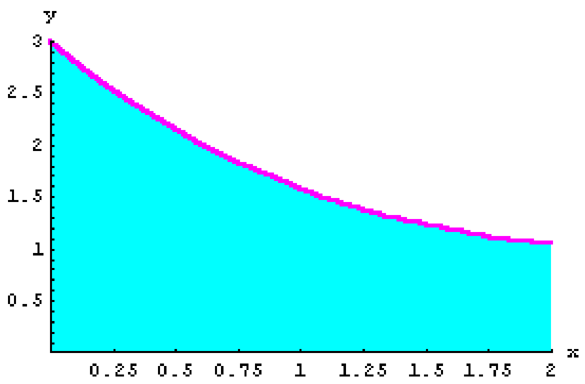
Remark. When the step size is reduced by a factor of $\frac{1}{2}$ the remainder term $E_{3c}(f, h)$ should be reduced by approximately $\left(\frac{1}{2}\right)^4 = 0.0625$.

Example 1. Let f[x] be $\int_0^2 \left(2 + \mathrm{Cos}\left[2\sqrt{x}\right]\right) dx$.

1 (a) Numerically approximate the integral by using Simpson's 3/8 rule with m = 1, 2, 4.
1 (b) Find the analytic value of the integral (i.e. find the "true value").
1 (c) Find the error for the Simpson' 3/8 rule approximations.
Solution 1.

Example 1. Let f[x] be $\int_0^2 (2 + \text{Cos}[2\sqrt{x}])\,dx$.

1 (a) Numerically approximate the integral by using Simpson's 3/8 rule with m = 1, 2, 4.
1 (b) Find the analytic value of the integral (i.e. find the "true value").
1 (c) Find the error for the Simpson' 3/8 rule approximations.
Solution 1 (a).



$f[x] = 2 + \text{Cos}[2\sqrt{x}]$

We will use simulated hand computations for the solution.

$$\mathbf{f[x\_] = 2 + Cos[2\sqrt{x}];}$$

$$\mathbf{s1} = \frac{3\,\frac{2-0}{3}}{8}\left(f[0] + 3\,f\left[\frac{2}{3}\right] + 3\,f\left[\frac{4}{3}\right] + f[2]\right)$$

NumberForm[N[s1], 12]

$$\frac{1}{4}\left(5 + 3\left(2 + \text{Cos}\left[2\sqrt{\frac{2}{3}}\right]\right) + \text{Cos}[2\sqrt{2}] + 3\left(2 + \text{Cos}\left[\frac{4}{\sqrt{3}}\right]\right)\right)$$

3.46059898098

$$\mathbf{s2} = \frac{3\,\frac{2-0}{6}}{8}\left(f[0] + 3\,f\left[\frac{1}{3}\right] + 3\,f\left[\frac{2}{3}\right] + 2\,f[1] + 3\,f\left[\frac{4}{3}\right] + 3\,f\left[\frac{5}{3}\right] + f[2]\right)$$

NumberForm[N[s2] , 12]

$$\frac{1}{8}\left(5 + 2\,(2 + \text{Cos}[2]) + 3\left(2 + \text{Cos}\left[2\sqrt{\frac{2}{3}}\right]\right) + 3\left(2 + \text{Cos}\left[2\sqrt{\frac{5}{3}}\right]\right) + \text{Cos}[2\sqrt{2}] + 3\left(2 + \text{Cos}\left[\frac{2}{\sqrt{3}}\right]\right) + 3\left(2 + \text{Cos}\left[\frac{4}{\sqrt{3}}\right]\right)\right)$$

3.46003538318

$$s4 = \frac{3\ \frac{2-0}{12}}{8}\left(f[0] + 3\ f\left[\frac{1}{6}\right] + 3\ f\left[\frac{1}{3}\right] + 2\ f\left[\frac{1}{2}\right] + 3\ f\left[\frac{2}{3}\right] + 3\ f\left[\frac{5}{6}\right] + 2\ f[1] + 3\ f\left[\frac{7}{6}\right] + 3\ f\left[\frac{4}{3}\right] + 2\ f\left[\frac{3}{2}\right] + 3\ f\left[\frac{5}{3}\right] + 3\ f\left[\frac{11}{6}\right] + f[2]\right)$$

NumberForm[N[s4], 12]

$$\frac{1}{16}\left(5 + 2\ (2 + \text{Cos}[2]) + 3\left(2 + \text{Cos}\left[\sqrt{\frac{2}{3}}\right]\right) + 3\left(2 + \text{Cos}\left[2\sqrt{\frac{2}{3}}\right]\right) + 3\left(2 + \text{Cos}\left[2\sqrt{\frac{5}{3}}\right]\right) + 2\ \left(2 + \text{Cos}\left[\sqrt{2}\right]\right) + \text{Cos}\left[2\sqrt{2}\right] + 3\left(2 + \text{Cos}\left[\frac{2}{\sqrt{3}}\right]\right) + 3\left(2 + \text{Cos}\left[\frac{4}{\sqrt{3}}\right]\right) + 3\left(2 + \text{Cos}\left[\sqrt{\frac{10}{3}}\right]\right) + 3\left(2 + \text{Cos}\left[\sqrt{\frac{14}{3}}\right]\right) + 2\ \left(2 + \text{Cos}\left[\sqrt{6}\right]\right) + 3\left(2 + \text{Cos}\left[\sqrt{\frac{22}{3}}\right]\right)\right)$$

3.46000003113

Solution 1 (b).

The integral of $f[x] = 2 + \text{Cos}\left[2\sqrt{x}\right]$ can be determined.

$$\int \left(2 + \text{Cos}\left[2\sqrt{x}\right]\right) dx$$

$$2\,x + \frac{1}{2}\text{Cos}\left[2\sqrt{x}\right] + \sqrt{x}\ \text{Sin}\left[2\sqrt{x}\right]$$

The value of the definite integral

$$val = \int_0^2 \left(2 + \text{Cos}\left[2\sqrt{x}\right]\right) dx$$

$$\frac{7}{2} + \frac{1}{2}\text{Cos}\left[2\sqrt{2}\right] + \sqrt{2}\ \text{Sin}\left[2\sqrt{2}\right]$$

N[val]

3.46

NumberForm[N[val] , 17]

3.459997672170804

Solution 1 (c).

val - t16

-0.000002358959196