1. Riemann Sums, Midpoint Rule and Trapezoidal Rule

Definition. Definite Integral as a Limit of a Riemann Sum. Let $f[x]$ be continuous over the interval $[a, b]$,
and let $P: a = x_0 < x_1 < x_2 < ... < x_n = b$ be a partition, then the definite integral is given by

$$\int_a^b f(x)\, dx = \lim \sum_{k=1}^{n} f[t_k]\, \Delta x_k ,$$

where $t_k \in [x_{k-1}, x_k]$ and the mesh size of the partition goes to zero in the "limit," i.e . $\Delta x_k \to 0$ as $n \to \infty$.

The midpoint rule uses $t_k = \frac{1}{2}(x_{k-1} + x_k)$ in the definition.

Improvements can be made in two directions, the **midpoint rule** evaluates the function at c_k, which is the midpoint of the subinterval $[x_k, x_{k+1}]$, i.e. $t_k = \frac{1}{2}(x_{k-1} + x_k)$ in the Riemann sum.

```
MidPointRule[a0_, b0_, n0_] :=
 Module[{a = a0, b = b0, c, ΔX, k, n = n0, X},
  ΔX = (b - a)/n;
  c[k_] = a + (k - 1/2) ΔX;
  Return[ Sum[f[c[k]] ΔX, {k, 1, n}]; ];
```

The Trapezoidal Rule is the average of the left Riemann sum and the right Riemann sum.

```
TrapezoidalRule[a0_, b0_, n0_] :=
 Module[{a = a0, b = b0, ΔX, k, n = n0, X},
  ΔX = (b - a)/n;
  X[k_] = a + k ΔX;
  Return[ Sum[((f[X[k-1]] + f[X[k]])/2) ΔX, {k, 1, n}]; ];
```

Example 1. Let $f[x] = x^2$ over $[0, 1]$.

1 (a) Find the formula for the left Riemann sum using n subintervals.
1 (b) Find the limit of the left Riemann sum in part (a).
Solution 1.

Example 2. Let $f[x] = x^2$ over $[0, 1]$.
2 (a) Find the formula for the right Riemann sum using n subintervals.
2 (b) Find the limit of the right Riemann sum in part (a).
Solution 2.

Example 3. Let $f[x] = x^2$ over $[0, 1]$.
3 (a) Find the formula for the midpoint rule sum using n subintervals.
3 (b) Find the limit of the midpoint rule sum in part (a).
Solution 3.

Example 4. Let $f[x] = x^2$ over $[0, 1]$.
4 (a) Find the formula for the trapezoidal rule sum using n subintervals.
4 (b) Find the limit of the trapezoidal rule sum in part (a).
Solution 4.

Example 5. Let $f[x] = x^2$ over $[0, 1]$. Compare the left Riemann sum, right Riemann sum, midpoint rule and trapezoidal rule for n = 100 subintervals. Compare them with the analytic solution.
Solution 5.

Example 1. Let $f[x] = x^2$ over $[0, 1]$.
1 (a) Find the formula for the left Riemann sum using n subintervals.
1 (b) Find the limit of the left Riemann sum in part (a).

Solution 1 (a).
Find the formula for the left Riemann sum using n subintervals.

$f[x] = x^2$ over $[0,1]$ using n subintervals.

$$x_{k-1} = \frac{-1+k}{n}$$

The Left Riemann Sum is:

$$\sum_{k=1}^{n} f[x_{k-1}]\Delta x = \sum_{k=1}^{n} \frac{(-1+k)^2}{n^2}\frac{1}{n}$$

$$\sum_{k=1}^{n} f[x_{k-1}]\Delta x = \sum_{k=1}^{n} \frac{(-1+k)^2}{n^3}$$

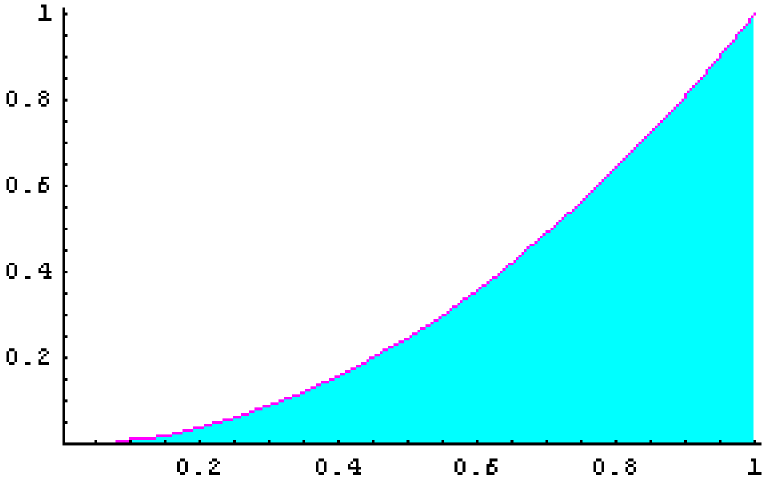
$$\sum_{k=1}^{n} f[x_{k-1}]\Delta x = \frac{1}{n^3}\sum_{k=1}^{n} (-1+k)^2$$

Use Fact 2: $\sum_{k=1}^{n} (k-1)^2 == \frac{1}{6}(-1+n)\,n\,(-1+2n)$

$$\sum_{k=1}^{n} f[x_{k-1}]\Delta x = \left(\frac{1}{6}(-1+n)\,n\,(-1+2n)\right)/(n^3)$$

$$\sum_{k=1}^{n} f[x_{k-1}]\Delta x = \frac{(-1+n)(-1+2n)}{6n^2}$$

Solution 1 (b).
Find the limit of the left Riemann sum in part (a).



f[x] = x^2 over [0,1].

The limit of the Left Riemann Sums is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_{k-1}]\Delta x = \lim_{n\to\infty} \left(\frac{1}{6}(-1+n)\,n\,(-1+2n)\right)/(n^3)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_{k-1}]\Delta x = \lim_{n\to\infty} \frac{(-1+n)(-1+2n)}{6n^2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_{k-1}]\Delta x = \lim_{n\to\infty} \frac{1-3n+2n^2}{6n^2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_{k-1}]\Delta x = \frac{1}{3}$$

Example 2. Let $f[x] = x^2$ over $[0, 1]$.
2 (a) Find the formula for the right Riemann sum using n subintervals.
2 (b) Find the limit of the right Riemann sum in part (a).

Solution 2 (a).
Find the formula for the right Riemann sum using n subintervals.

$f[x] = x^2$ over $[0,1]$ using n subintervals.

$$x_{k-1} = \frac{-1+k}{n}$$

The Right Riemann Sum is:

$$\sum_{k=1}^{n} f[x_k]\Delta x = \sum_{k=1}^{n} \frac{k^2}{n^2}\frac{1}{n}$$

$$\sum_{k=1}^{n} f[x_k]\Delta x = \sum_{k=1}^{n} \frac{k^2}{n^3}$$

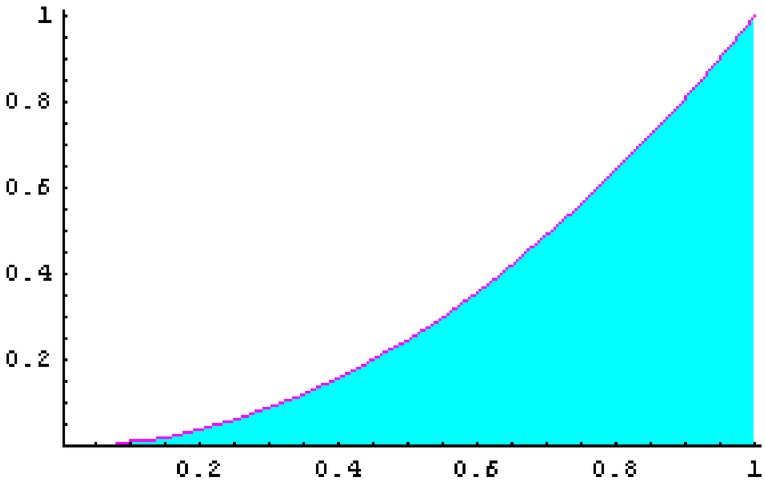
$$\sum_{k=1}^{n} f[x_k]\Delta x = \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

Use Fact 2: $\displaystyle\sum_{k=1}^{n} k^2 == \frac{1}{6} n (1+n)(1+2n)$

$$\sum_{k=1}^{n} f[x_k]\Delta x = (\frac{1}{6} n (1+n)(1+2n))/(n^3)$$

$$\sum_{k=1}^{n} f[x_k]\Delta x = \frac{(1+n)(1+2n)}{6n^2}$$

Solution 2 (b).
Find the limit of the right Riemann sum in part (a).



f[x] = x^2 over [0,1].

The limit of the Right Riemann Sums is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_{k-1}]\Delta x = \lim_{n\to\infty} \left(\frac{1}{6} n (1+n)(1+2n)\right)/(n^3)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_k]\Delta x = \lim_{n\to\infty} \frac{(1+n)(1+2n)}{6n^2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_k]\Delta x = \lim_{n\to\infty} \frac{1+3n+2n^2}{6n^2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_k]\Delta x = \frac{1}{3}$$

The area under the curve $y = x^2$ is $\frac{1}{3}$

Example 3. Let $f[x] = x^2$ over $[0, 1]$.

3 (a) Find the formula for the midpoint rule sum using n subintervals.

3 (b) Find the limit of the midpoint rule sum in part (a).

Solution 3 (a).

Find the formula for the midpoint rule sum using n subintervals.

$f[x] = x^2$ over $[0,1]$ using n subintervals.

$$t_k = \frac{\frac{1}{2} + k}{n}$$

The Midpoint Rule is:

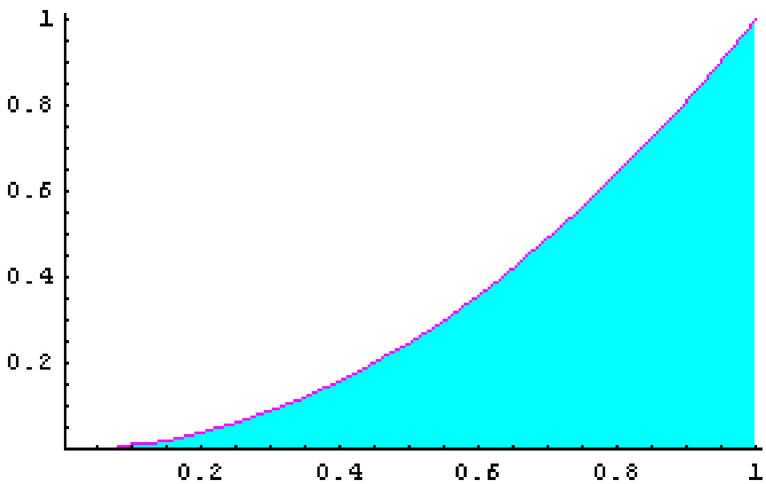
$$\sum_{k=1}^{n} f[t_k]\Delta x = \sum_{k=1}^{n} \frac{\left(\frac{1}{2} + k\right)^2}{n^2} \frac{1}{n}$$

$$\sum_{k=1}^{n} f[t_k]\Delta x = \sum_{k=1}^{n} \frac{\left(\frac{1}{2} + k\right)^2}{n^3}$$

$$\sum_{k=1}^{n} f[t_k]\Delta x = \frac{(-1 + 2n)(1 + 2n)}{12 n^2}$$

Solution 3 (b).

Find the limit of the midpoint rule sum in part (a).



$f[x] = x^2$ over $[0,1]$.

The limit of the Midpoint Rule is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_k]\Delta x = \lim_{n\to\infty} \frac{(-1 + 2n)(1 + 2n)}{12 n^2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_k]\Delta x = \lim_{n\to\infty} \frac{-1+4n^2}{12n^2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_k]\Delta x = \frac{1}{3}$$

The area under the curve $y = x^2$ is $\frac{1}{3}$

Example 4. Let $f[x] = x^2$ over $[0, 1]$.
4 (a) Find the formula for the trapezoidal rule sum using n subintervals.
4 (b) Find the limit of the trapezoidal rule sum in part (a).

Solution 4 (a).
Find the formula for the trapezoidal rule sum using n subintervals.

$f[x] = x^2$ over $[0,1]$ using n subintervals.

$$x_k = \frac{k}{n}$$

The Trapezoidal Rule is:

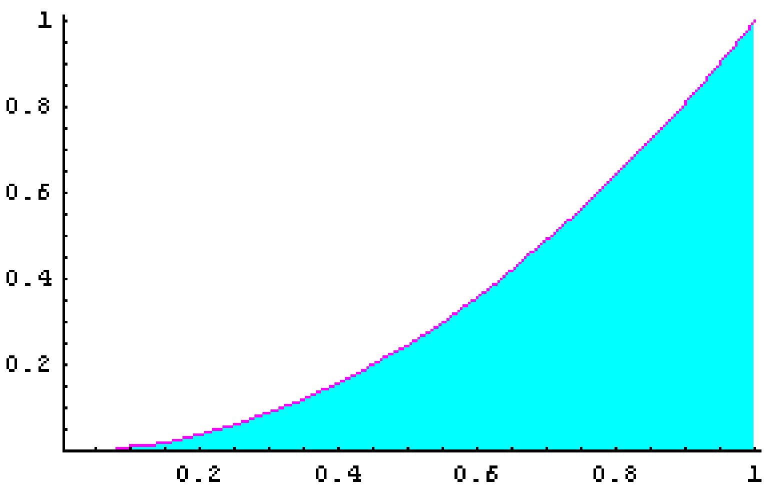
$$\sum_{k=0}^{-1+n} \left(\frac{f[x_k] + f[x_{k+1}]}{2}\right) \Delta x = \sum_{k=0}^{-1+n} \frac{1}{2} \left(\frac{k^2}{n^2} + \frac{(1+k)^2}{n^2}\right) \frac{1}{n}$$

$$\sum_{k=0}^{-1+n} \left(\frac{f[x_k] + f[x_{k+1}]}{2}\right)] \Delta x = \sum_{k=0}^{-1+n} \frac{\frac{k^2}{n^2} + \frac{(1+k)^2}{n^2}}{2\,n}$$

$$\sum_{k=0}^{-1+n} \left(\frac{f[x_k] + f[x_{k+1}]}{2}\right)] \Delta x = \sum_{k=0}^{-1+n} \frac{1 + 2\,k + 2\,k^2}{2\,n^3}$$

$$\sum_{k=0}^{-1+n} \left(\frac{f[x_k] + f[x_{k+1}]}{2}\right) \Delta x = \frac{1 + 2\,n^2}{6\,n^2}$$

Solution 4 (b).
Find the limit of the trapezoidal rule sum in part (a).



$f[x] = x^2$ over $[0,1]$.
The limit of the Trapezoidal Rule is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_k]\Delta x = \lim_{n\to\infty} \frac{1+2n^2}{6n^2}$$

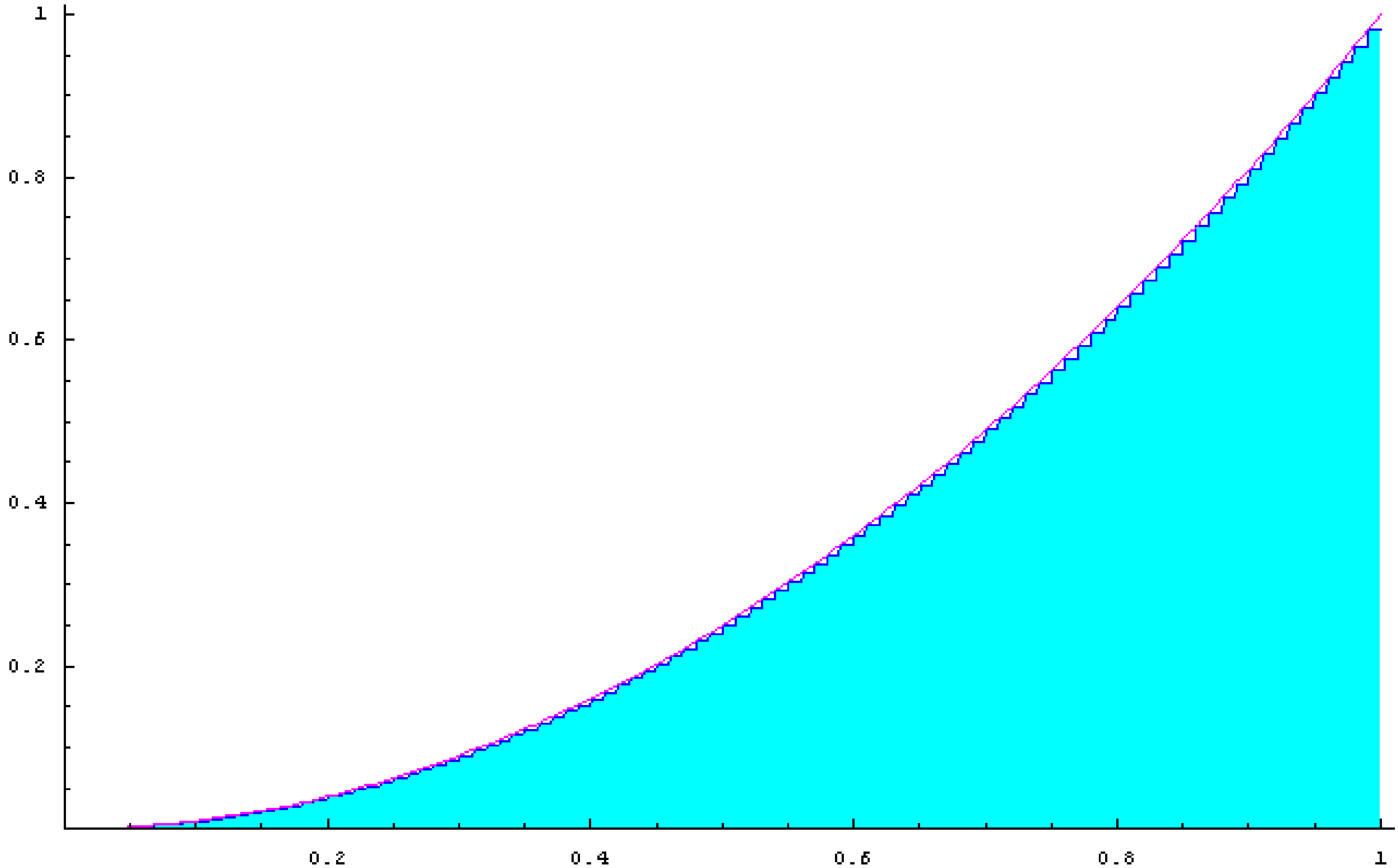
$$\lim_{n\to\infty} \sum_{k=1}^{n} f[x_k]\Delta x = \frac{1}{3}$$

The area under the curve $y = x^2$ is $\frac{1}{3}$

Example 5. Let $f[x] = x^2$ over $[0, 1]$. Compare the left Riemann sum, right Riemann sum, midpoint rule and trapezoidal rule for n = 100 subintervals. Compare them with the analytic solution.

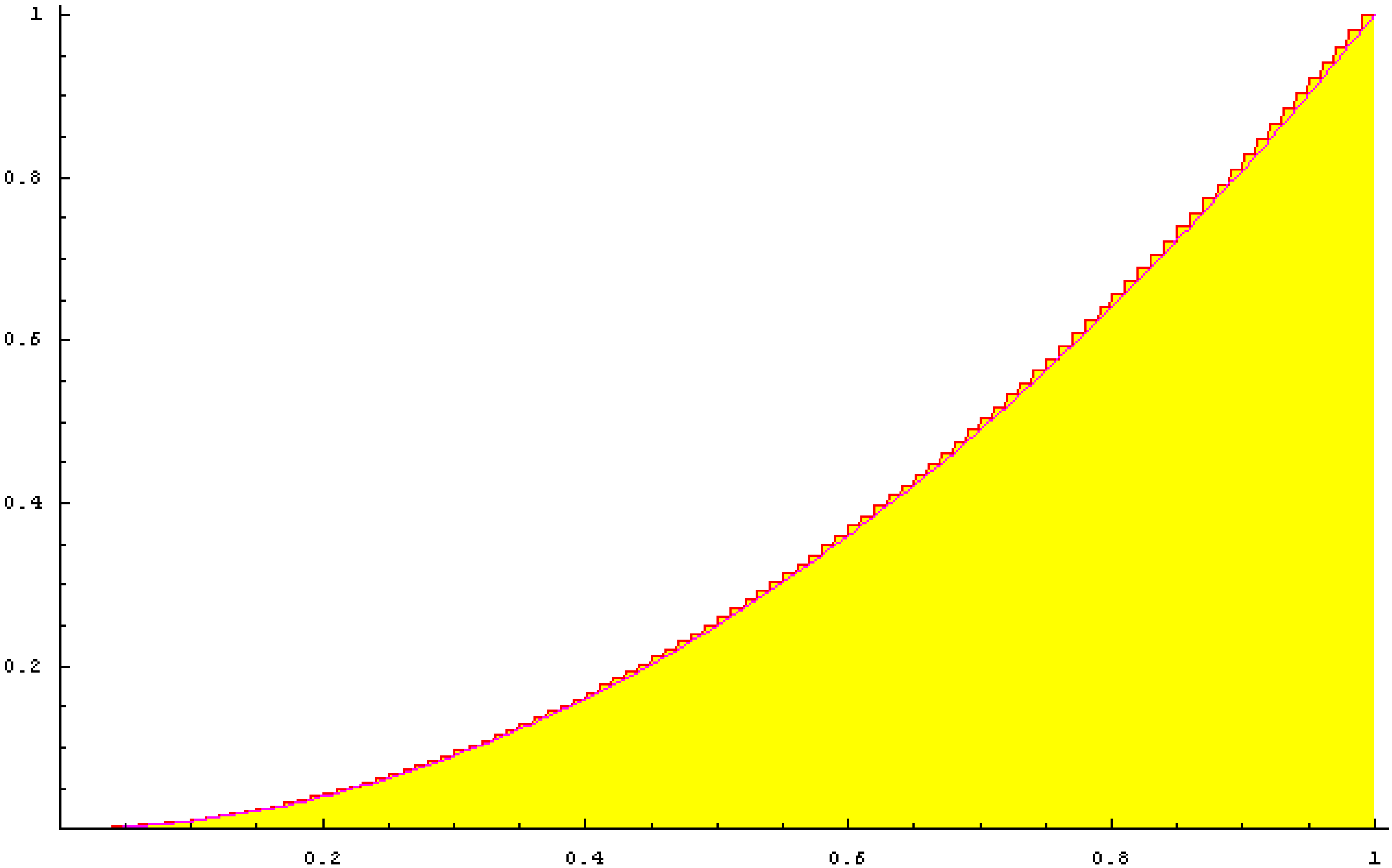
Solution 5.

$f[x] = x^2$ over [0,1] using 100 subintervals.



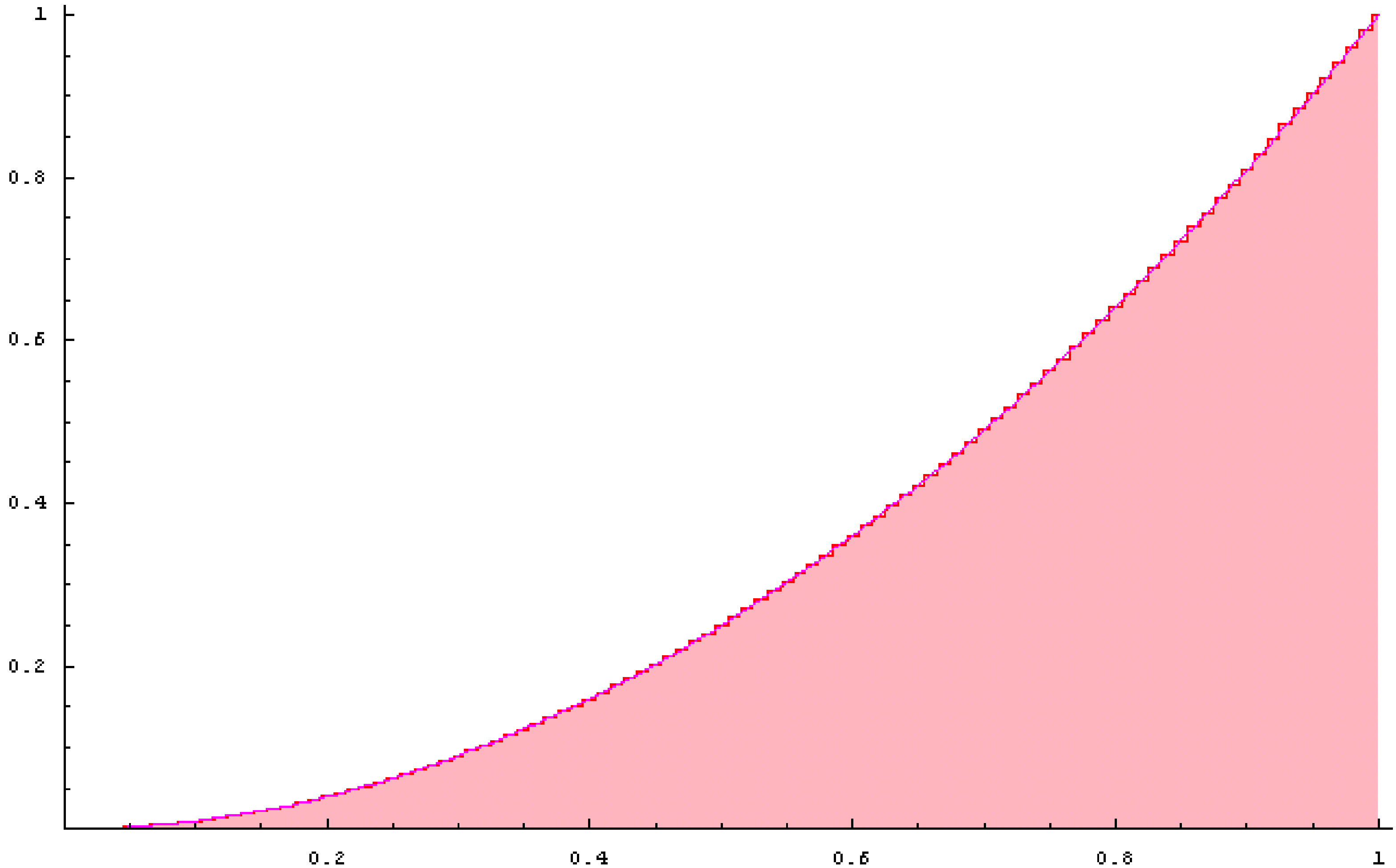
The Left Riemann Sum is:

$$\sum_{k=1}^{100} f[x_{k-1}]\Delta x = \frac{6567}{20000} = 0.32835$$



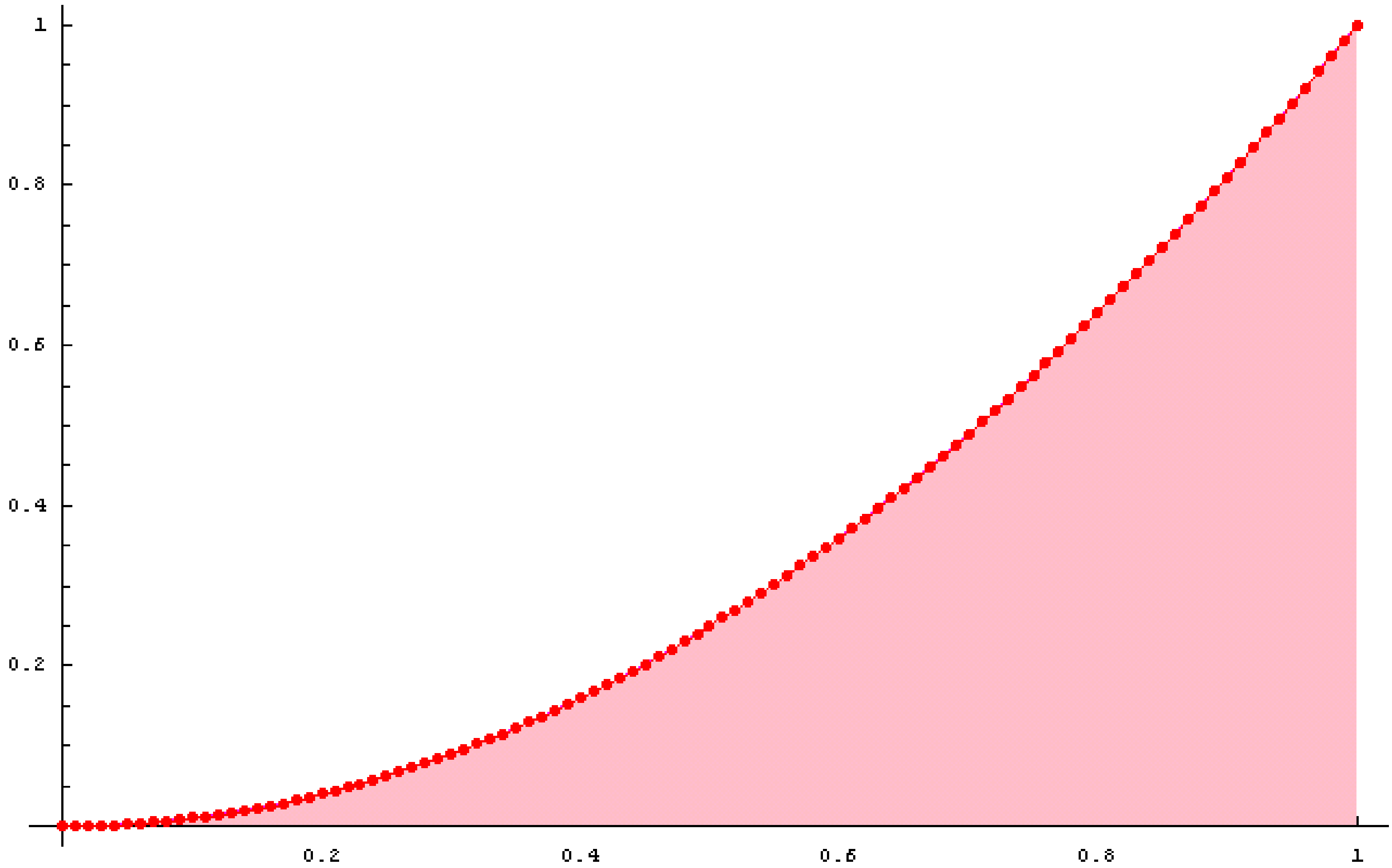
The Right Riemann Sum is:

$$\sum_{k=1}^{100} f[x_k]\Delta x = \frac{6767}{20000} = 0.33835$$



The Midpoint Rule is:

$$\sum_{k=0}^{99} f[x_k]\Delta x = \frac{13333}{40000} = 0.333325$$



The Trapezoidal Rule is:

$$\sum_{k=0}^{99} f[x_k]\Delta x = \frac{6667}{20000} = 0.33335$$

As a result,

$$\text{Left Riemann Sum} = \frac{6567}{20000} = 0.32835$$

$$\text{Right Riemann Sum} = \frac{6767}{20000} = 0.33835$$

$$\text{`Average` of Riemann Sums} = \frac{6667}{20000} = 0.33335$$

$$\text{Midpoint Rule} = \frac{13333}{40000} = 0.333325$$

$$\text{Trapezoidal Rule} = \frac{6667}{20000} = 0.33335$$

$$\text{Analytic Value} = \frac{1}{3} = 0.3333333333333333$$

Observe that the Trapezoidal Rule is the `Average` of Riemann Sums.