2. The Midpoint Rule for Numerical Integration

Theorem (Midpoint Rule) Consider $y = f(x)$ over $[x_0, x_1]$, where $x_1 = x_0 + h$. The midpoint rule is

$$MP(f, h) = h f\left(x_0 + \frac{h}{2}\right).$$

This is an numerical approximation to the integral of $f(x)$ over $[x_0, x_1]$ and we have the expression

$$\int_{x_0}^{x_1} f(x)\,dx \approx MP(f, h).$$

The remainder term for the midpoint rule is $R_{MP}(f, h) = \frac{1}{24} f''(c) h^3$, where c lies somewhere between x_0 and x_1, and have the equality

$$\int_{x_0}^{x_1} f(x)\,dx = h f\left(x_0 + \frac{h}{2}\right) + \frac{1}{24} f''(c) h^3.$$

Composite Midpoint Rule

An intuitive method of finding the area under a curve y = f(x) is by approximating that area with a series of rectangles that lie above the intervals $\{[x_{k-1}, x_k]\}_{k=1}^{m}$. When several rectangles are used, we call it the composite midpoint rule.

Theorem (Composite Midpoint Rule) Consider $y = f(x)$ over $[a, b]$. Suppose that the interval $[a, b]$ is subdivided into m subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{m}$ of equal width $h = \frac{b - a}{m}$ by using the equally spaced nodes $c_k = a + \left(k - \frac{1}{2}\right) h$ for $k = 1, 2, \ldots, m$. The composite midpoint rule for m subintervals is

$$M(f, h) = h \sum_{k=1}^{m} f(c_k).$$

This is an numerical approximation to the integral of $f(x)$ over $[a, b]$ and we write

$$\int_a^b f(x)\,dx \approx M(f, h).$$

Remainder term for the Composite Midpoit Rule

Corollary (Midpoint Rule: Remainder term) Suppose that $[a, b]$ is subdivided into m subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{m}$ of width $h = \frac{b-a}{m}$. The composite midpoint rule

$$M(f, h) = h \sum_{k=1}^{m} f\left(a + \left(k - \frac{1}{2}\right) h\right)$$

is an numerical approximation to the integral, and

$$\int_a^b f(x)\, dx = M(f, h) + E_M(f, h).$$

Furthermore, if $f(x) \in C^2[a, b]$, then there exists a value c with a < c < b so that the error term $E_M(f, h)$ has the form

$$E_M(f, h) = \frac{(b-a) f^{2}(c)}{24} h^2.$$

This is expressed using the "big O" notation $E_M(f, h) = O(h^2)$.

Remark. When the step size is reduced by a factor of $\frac{1}{2}$ the error term $E_M(f, h)$ should be reduced by approximately $\left(\frac{1}{2}\right)^2 = 0.25$.

Example 1. Consider the integral $\int_0^2 \left(2 + \text{Cos}\left[2\sqrt{x}\right]\right) dx$.

1 (a) Numerically approximate the integral by using the midpoint rule with m = 1, 2, 4, 8, and 16 subintervals.

1 (b) Find the analytic value of the integral (i.e. find the "true value").

Solution 1.

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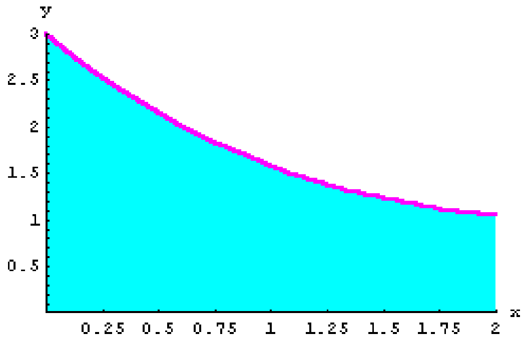
Solution 1 (a).



$f[x] = 2 + \text{Cos}\left[2\sqrt{x}\right]$

We will use simulated hand computations for the solution.

$$\mathbf{f[x\_] = 2 + Cos\left[2\sqrt{x}\right];}$$

$$\mathbf{m1 = \frac{2-0}{1}\,(f[1])}$$

N[m1]

$2\,(2 + \text{Cos}[2])$

3.16771

$$\mathbf{m2 = \frac{2-0}{2}\left(f\left[\frac{1}{2}\right] + f\left[\frac{3}{2}\right]\right)}$$

N[m2]

$4 + \text{Cos}\left[\sqrt{2}\right] + \text{Cos}\left[\sqrt{6}\right]$

3.38604

$$\mathbf{m4 = \frac{2-0}{4}\left(f\left[\frac{1}{4}\right] + f\left[\frac{3}{4}\right] + f\left[\frac{5}{4}\right] + f\left[\frac{7}{4}\right]\right)}$$

N[m4]

$\frac{1}{2}\left(8 + \text{Cos}[1] + \text{Cos}\left[\sqrt{3}\right] + \text{Cos}\left[\sqrt{5}\right] + \text{Cos}\left[\sqrt{7}\right]\right)$

3.44145

$$\text{m8} = \frac{2-0}{8}\left(f\left[\frac{1}{8}\right]+f\left[\frac{3}{8}\right]+f\left[\frac{5}{8}\right]+f\left[\frac{7}{8}\right]+f\left[\frac{9}{8}\right]+f\left[\frac{11}{8}\right]+f\left[\frac{13}{8}\right]+f\left[\frac{15}{8}\right]\right)$$

N[m8]

$$\frac{1}{4}\left(16+\text{Cos}\left[\sqrt{\frac{3}{2}}\right]+\text{Cos}\left[\frac{1}{\sqrt{2}}\right]+\text{Cos}\left[\frac{3}{\sqrt{2}}\right]+\text{Cos}\left[\sqrt{\frac{5}{2}}\right]+\text{Cos}\left[\sqrt{\frac{7}{2}}\right]+\text{Cos}\left[\sqrt{\frac{11}{2}}\right]+\text{Cos}\left[\sqrt{\frac{13}{2}}\right]+\text{Cos}\left[\sqrt{\frac{15}{2}}\right]\right)$$

3.45536

$$\text{m16} = \frac{2-0}{16}\left(f\left[\frac{1}{16}\right]+f\left[\frac{3}{16}\right]+f\left[\frac{5}{16}\right]+f\left[\frac{7}{16}\right]+f\left[\frac{9}{16}\right]+f\left[\frac{11}{16}\right]+f\left[\frac{13}{16}\right]+f\left[\frac{15}{16}\right]+f\left[\frac{17}{16}\right]+f\left[\frac{19}{16}\right]+f\left[\frac{21}{16}\right]+f\left[\frac{23}{16}\right]+f\left[\frac{25}{16}\right]+f\left[\frac{27}{16}\right]+f\left[\frac{29}{16}\right]+f\left[\frac{31}{16}\right]\right)$$

N[m16]

$$\frac{1}{8}\left(32+\text{Cos}\left[\frac{1}{2}\right]+\text{Cos}\left[\frac{3}{2}\right]+\text{Cos}\left[\frac{5}{2}\right]+\text{Cos}\left[\frac{\sqrt{3}}{2}\right]+\text{Cos}\left[\frac{3\sqrt{3}}{2}\right]+\text{Cos}\left[\frac{\sqrt{5}}{2}\right]+\right.$$
$$\left.\text{Cos}\left[\frac{\sqrt{7}}{2}\right]+\text{Cos}\left[\frac{\sqrt{11}}{2}\right]+\text{Cos}\left[\frac{\sqrt{13}}{2}\right]+\text{Cos}\left[\frac{\sqrt{15}}{2}\right]+\text{Cos}\left[\frac{\sqrt{17}}{2}\right]+\text{Cos}\left[\frac{\sqrt{19}}{2}\right]+\text{Cos}\left[\frac{\sqrt{21}}{2}\right]+\text{Cos}\left[\frac{\sqrt{23}}{2}\right]+\text{Cos}\left[\frac{\sqrt{29}}{2}\right]+\text{Cos}\left[\frac{\sqrt{31}}{2}\right]\right)$$

3.45884

Solution 1 (b).

$$\text{val} = \int_0^2 \left(2+\text{Cos}\left[2\sqrt{x}\right]\right)\,dx$$

$$\frac{7}{2}+\frac{1}{2}\text{Cos}\left[2\sqrt{2}\right]+\sqrt{2}\,\text{Sin}\left[2\sqrt{2}\right]$$

N[val]

3.46

NumberForm[N[val] , 12]

3.45999767217