3. Derivation of Numerical Differentiation Formulae

The traditional "pencil and paper" derivations for numerical differentiation formulas for $f'[x_0]$ and $f''[x_0]$ are done independently as if there was no connection among the derivations. This new approach gives a parallel development of the formulas. It requires the solution of a "linear system" that includes symbolic quantities as coefficients and constants. The power of a computer algebra system such as *Mathematica* is used to elegantly solve the linear system for $f'[x_0]$ and $f''[x_0]$.

The Three Point Central Difference Formulas

Using three points $f[x_0-h]$, $f[x_0]$, and $f[x_0+h]$ we give a parallel development of the numerical differentiation formulas for $f'[x_0]$ and $f''[x_0]$. Start with the Taylor series for $f[x_0+h]$ expanded in powers of h. Terms must be included so that the remainder term is included. Using degree $n=4$ will suffice.

$$f[x_0+h] = f[x_0] + f'[x_0]\,h + \frac{1}{2} f''[x_0]\,h^2 + \frac{1}{6} f^{(3)}[x_0]\,h^3 + \frac{1}{24} f^{(4)}[x_0]\,h^4 + O[h]^5$$

The accuracy for the Taylor polynomial is $O[h]^5$.

$$f[x_0+h] \approx f[x_0] + h\,f'[x_0] + \frac{1}{2} h^2 f''[x_0] + \frac{1}{6} h^3 f^{(3)}[x_0] + \frac{1}{24} h^4 f^{(4)}[x_0]$$

Construct two "equations" by setting the series equal to the function at $x = x_0 \pm h$.

$$f[x_0] - h\,f'[x_0] + \tfrac{1}{2} h^2 f''[x_0] - \tfrac{1}{6} h^3 f^{(3)}[x_0] + \tfrac{1}{24} h^4 f^{(4)}[x_0] == f[-h+x_0]$$
$$f[x_0] + h\,f'[x_0] + \tfrac{1}{2} h^2 f''[x_0] + \tfrac{1}{6} h^3 f^{(3)}[x_0] + \tfrac{1}{24} h^4 f^{(4)}[x_0] == f[h+x_0]$$

We have two equations in the two unknown $f'[x_0]$ and $f''[x_0]$, and all the other quantities h, $f[x_0-h]$, $f[x_0]$, $f[x_0+h]$, $f^{(3)}[x_0]$ and $f^{(4)}[x_0]$ are considered as constants.

Solve the equations

$$f[x_0] - h\,f'[x_0] + \tfrac{1}{2} h^2 f''[x_0] - \tfrac{1}{6} h^3 f^{(3)}[x_0] + \tfrac{1}{24} h^4 f^{(4)}[x_0] == f[-h+x_0]$$
$$f[x_0] + h\,f'[x_0] + \tfrac{1}{2} h^2 f''[x_0] + \tfrac{1}{6} h^3 f^{(3)}[x_0] + \tfrac{1}{24} h^4 f^{(4)}[x_0] == f[h+x_0]$$

Get

$$f'[x_0] \to -\frac{3 f[-h+x_0] - 3 f[h+x_0] + h^3 f^{(3)}[x_0]}{6h}$$
$$f''[x_0] \to -\frac{24 f[x_0] - 12 f[-h+x_0] - 12 f[h+x_0] + h^4 f^{(4)}[x_0]}{12 h^2}$$

The result is not too easy to read, so we can use manipulate the formula.

$$f'[x_0] = \frac{-f[-h+x_0] + f[h+x_0]}{2h} - \frac{1}{6} h^2 f^{(3)}[x_0] + \ldots$$
$$f''[x_0] = \frac{-2 f[x_0] + f[-h+x_0] + f[h+x_0]}{h^2} - \frac{1}{12} h^2 f^{(4)}[x_0] + \ldots$$

Thus, we have derived the numerical differentiation formula for $f'[x_0]$ and $f''[x_0]$ and the first term in the series expansion for the remainder which involves, $f^{(3)}[x_0]$ or $f^{(4)}[x_0]$, respectively. Since the "numerical differentiation formulae" are "truncated" infinite series, we know that the ellipses "$\ldots$" means that there are infinitely many more term which are not shown.

When we do not include the "$\ldots$", we must evaluate the lowest order derivative in the series for the remainder at the value $x = c$ instead of $x = x_0$, then we can "chop off" the infinite series at the term involving $f^{(3)}[c]$ or $f^{(4)}[c]$, respectively.

$$f'[x_0] = \frac{-f[-h+x_0]+f[h+x_0]}{2h} - \frac{1}{6}h^2 f^{(3)}[c]$$
$$f''[x_0] = \frac{-2f[x_0]+f[-h+x_0]+f[h+x_0]}{h^2} - \frac{1}{12}h^2 f^{(4)}[c]$$

We have obtained the desired numerical differentiation formulas and their remainder terms. We can add the term $O[h]^2$ to the numerical differentiation formula term

$$\frac{-f[-h+x_0]+f[h+x_0]}{2h} + O[h]^2 = f'[x_0] + O[h]^2$$

$$\frac{-2f[x_0]+f[-h+x_0]+f[h+x_0]}{h^2} + O[h]^2 = f''[x_0] + O[h]^2$$

Therefore, we have established the numerical differentiation formulas

$$f'[x_0] = \frac{f[x_0+h] - f[x_0-h]}{2h} - \frac{1}{6}f^{(3)}[c]\,h^2$$

and

$$f''[x_0] = \frac{f[x_0-h] - 2f[x_0] + f[x_0+h]}{h^2} - \frac{1}{12}f^{(4)}[c]\,h^2$$

And the corresponding formulas with the big "O" notation $O[h]^2$ are

$$f'[x_0] = \frac{f[x_0+h] - f[x_0-h]}{2h} + O[h]^2$$

and

$$f''[x_0] = \frac{f[x_0-h] - 2f[x_0] + f[x_0+h]}{h^2} + O[h]^2$$

Comparison with the Traditional Derivations

The above derivation differs only slightly from the traditional derivations which also start with the two equations

$$f[x_0-h] = f[x_0] - h f'[x_0] + \frac{1}{2}h^2 f''[x_0] - \frac{1}{6}h^3 f^{(3)}[x_0] + \frac{1}{24}h^4 f^{(4)}[x_0] + \ldots$$

(1)

$$f[h+x_0] = f[x_0] + h f'[x_0] + \frac{1}{2}h^2 f''[x_0] + \frac{1}{6}h^3 f^{(3)}[x_0] + \frac{1}{24}h^4 f^{(4)}[x_0] + \ldots$$

When deriving the numerical differentiation formula for $f'[x_0]$ the equations (1) are subtracted and the terms involving $f''[x_0]$ cancel and manipulations are used to solve for $f'[x_0]$ and its truncation error term.

The traditional derivation of the numerical differentiation formula for $f''[x_0]$ starts with the same equations (1). But this time the equations are added and the terms involving $f'[x_0]$ cancel and manipulations are used to solve for $f''[x_0]$ and its truncation error term.

There is no "big deal" made about the fact that the starting place is the same. We shall see for the higher order formulas that using the same starting place will be the key to successful computer derivations of numerical differentiation formulas.

The Five Point Central Difference Formulas

Using five points $f[x_0-2h]$, $f[x_0-h]$, $f[x_0]$, $f[x_0+h]$, and $f[x_0+2h]$ we can give a parallel development of the numerical differentiation formulas for $f'[x_0]$, $f''[x_0]$, $f^{(3)}[x_0]$ and $f^{(4)}[x_0]$. Start with the Taylor series for $f[x_0+h]$ expanded in powers of h. Terms must be included so that the remainder term is included. Using degree $n = 6$ will suffice.

$$f[x_0+h] \approx f[x_0] + h\,f'[x_0] + \frac{1}{2}h^2 f''[x_0] + \frac{1}{6}h^3 f^{(3)}[x_0] + \frac{1}{24}h^4 f^{(4)}[x_0] + \frac{1}{120}h^5 f^{(5)}[x_0] + \frac{1}{720}h^6 f^{(6)}[x_0]$$

Use five points and set up the "equations" by setting $s[x_0 + k\,h] == f[x_0 + k\,h]$ for $k = -2, -1, 0, 1, 2$. Although the value $k = 0$, will produce $f[x_0] == f[x_0]$ which is the boolean value True, it will do no harm when solving the set of equations.

$$f[x_0] - 2h\,f'[x_0] + 2h^2 f''[x_0] - \tfrac{4}{3}h^3 f^{(3)}[x_0] + \tfrac{2}{3}h^4 f^{(4)}[x_0] - \tfrac{4}{15}h^5 f^{(5)}[x_0] + \tfrac{4}{45}h^6 f^{(6)}[x_0] == f[-2h + x_0]$$

$$f[x_0] - h\,f'[x_0] + \tfrac{1}{2}h^2 f''[x_0] - \tfrac{1}{6}h^3 f^{(3)}[x_0] + \tfrac{1}{24}h^4 f^{(4)}[x_0] - \tfrac{1}{120}h^5 f^{(5)}[x_0] + \tfrac{1}{720}h^6 f^{(6)}[x_0] == f[-h + x_0]$$

True

$$f[x_0] + h\,f'[x_0] + \tfrac{1}{2}h^2 f''[x_0] + \tfrac{1}{6}h^3 f^{(3)}[x_0] + \tfrac{1}{24}h^4 f^{(4)}[x_0] + \tfrac{1}{120}h^5 f^{(5)}[x_0] + \tfrac{1}{720}h^6 f^{(6)}[x_0] == f[h + x_0]$$

$$f[x_0] + 2h\,f'[x_0] + 2h^2 f''[x_0] + \tfrac{4}{3}h^3 f^{(3)}[x_0] + \tfrac{2}{3}h^4 f^{(4)}[x_0] + \tfrac{4}{15}h^5 f^{(5)}[x_0] + \tfrac{4}{45}h^6 f^{(6)}[x_0] == f[2h + x_0]$$

We can consider that these are four equations in the four unknowns $f'[x_0]$, $f''[x_0]$, $f^{(3)}[x_0]$ and $f^{(4)}[x_0]$, and all the other quantities h, $f[x_0 - 2h]$, $f[x_0 - h]$, $f[x_0]$, $f[x_0 + h]$, $f[x_0 - 2h]$, $f^{(5)}[x_0]$ and $f^{(6)}[x_0]$ are constants. Since this requires typing of four derivatives to be solved, we will automate this process too, by using a table to construct the "variables."

The variables in the above equations are

$\{f'[x_0], f''[x_0], f^{(3)}[x_0], f^{(4)}[x_0]\}$

Amazingly, the algebra involved in solving the four equations will result in the cancellation of $f^{(6)}[x_0]$ in the odd order derivatives and the term $f^{(5)}[x_0]$ will cancel in the even order derivatives.

Solve the equations

$$f[x_0] - 2h\,f'[x_0] + 2h^2 f''[x_0] - \tfrac{4}{3}h^3 f^{(3)}[x_0] + \tfrac{2}{3}h^4 f^{(4)}[x_0] - \tfrac{4}{15}h^5 f^{(5)}[x_0] + \tfrac{4}{45}h^6 f^{(6)}[x_0] == f[-2h + x_0]$$

$$f[x_0] - h\,f'[x_0] + \tfrac{1}{2}h^2 f''[x_0] - \tfrac{1}{6}h^3 f^{(3)}[x_0] + \tfrac{1}{24}h^4 f^{(4)}[x_0] - \tfrac{1}{120}h^5 f^{(5)}[x_0] + \tfrac{1}{720}h^6 f^{(6)}[x_0] == f[-h + x_0]$$

True

$$f[x_0] + h\,f'[x_0] + \tfrac{1}{2}h^2 f''[x_0] + \tfrac{1}{6}h^3 f^{(3)}[x_0] + \tfrac{1}{24}h^4 f^{(4)}[x_0] + \tfrac{1}{120}h^5 f^{(5)}[x_0] + \tfrac{1}{720}h^6 f^{(6)}[x_0] == f[h + x_0]$$

$$f[x_0] + 2h\,f'[x_0] + 2h^2 f''[x_0] + \tfrac{4}{3}h^3 f^{(3)}[x_0] + \tfrac{2}{3}h^4 f^{(4)}[x_0] + \tfrac{4}{15}h^5 f^{(5)}[x_0] + \tfrac{4}{45}h^6 f^{(6)}[x_0] == f[2h + x_0]$$

Get

$$f'[x_0] \to -\frac{-5 f[-2h+x_0] + 40 f[-h+x_0] - 40 f[h+x_0] + 5 f[2h+x_0] - 2h^5 f^{(5)}[x_0]}{60\,h}$$

$$f''[x_0] \to -\frac{450 f[x_0] + 15 f[-2h+x_0] - 240 f[-h+x_0] - 240 f[h+x_0] + 15 f[2h+x_0] - 2h^6 f^{(6)}[x_0]}{180\,h^2}$$

$$f^{(3)}[x_0] \to -\frac{2 f[-2h+x_0] - 4 f[-h+x_0] + 4 f[h+x_0] - 2 f[2h+x_0] + h^5 f^{(5)}[x_0]}{4\,h^3}$$

$$f^{(4)}[x_0] \to -\frac{-36 f[x_0] - 6 f[-2h+x_0] + 24 f[-h+x_0] + 24 f[h+x_0] - 6 f[2h+x_0] + h^6 f^{(6)}[x_0]}{6\,h^4}$$

The output is not too clear to read, so we shall use the formula manipulation to group the numerical differentiation portion

$$f'[x_0] = \frac{f[-2h + x_0] - 8 f[-h + x_0] + 8 f[h + x_0] - f[2h + x_0]}{12\,h} + \frac{1}{30}h^4 f^{(5)}[c]$$

$$f''[x_0] = \frac{-30 f[x_0] - f[-2h + x_0] + 16 f[-h + x_0] + 16 f[h + x_0] - f[2h + x_0]}{12\,h^2} + \frac{1}{90}h^4 f^{(6)}[c]$$

$$f^{(3)}[x_0] = \frac{-f[-2h + x_0] + 2 f[-h + x_0] - 2 f[h + x_0] + f[2h + x_0]}{2\,h^3} - \frac{1}{4}h^2 f^{(5)}[c]$$

$$f^{(4)}[x_0] = \frac{6 f[x_0] + f[-2h + x_0] - 4 f[-h + x_0] - 4 f[h + x_0] + f[2h + x_0]}{h^4} - \frac{1}{6}h^2 f^{(6)}[c]$$

Notice that the formula for approximating $f'[x_0]$ and $f''[x_0]$ have truncation error terms involving h^4 so they are numerical differentiation formulas of order $O[h]^4$. But the formula for approximating $f^{(3)}[x_0]$ and $f^{(4)}[x_0]$ have truncation error terms involving h^2 so they are numerical differentiation formulas of order $O[h]^2$. This is one of the many surprises in the theory of numerical analysis.

$$\frac{f[-2h+x_0]-8f[-h+x_0]+8f[h+x_0]-f[2h+x_0]}{12h}+O[h]^4 = f'[x_0]+O[h]^4$$

$$\frac{-30f[x_0]-f[-2h+x_0]+16f[-h+x_0]+16f[h+x_0]-f[2h+x_0]}{12h^2}+O[h]^4 = f''[x_0]+O[h]^4$$

$$\frac{-f[-2h+x_0]+2f[-h+x_0]-2f[h+x_0]+f[2h+x_0]}{2h^3}+O[h]^2 = f^{(3)}[x_0]+O[h]^2$$

$$\frac{6f[x_0]+f[-2h+x_0]-4f[-h+x_0]-4f[h+x_0]+f[2h+x_0]}{h^4}+O[h]^2 = f^{(4)}[x_0]+O[h]^2$$

Therefore, we have established the numerical differentiation formulas

$$f'[x_0] = \frac{f[x_0-2h]-8f[x_0-h]+8f[x_0+h]-f[x_0+2h]}{12h}+\frac{1}{30}h^4 f^{(5)}[c]$$

$$f''[x_0] = \frac{-f[x_0-2h]+16f[x_0-h]-30f[x_0]+16f[x_0+h]-f[x_0+2h]}{12h^2}+\frac{1}{90}h^4 f^{(6)}[c]$$

$$f^{(3)}[x_0] = \frac{-f[x_0-2h]+2f[x_0-h]-2f[x_0+h]+f[x_0+2h]}{2h^3}-\frac{1}{4}h^2 f^{(5)}[c]$$

$$f^{(4)}[x_0] = \frac{f[x_0-2h]-4f[x_0-h]+6f[x_0]-4f[x_0+h]+f[x_0+2h]}{h^4}-\frac{1}{6}h^2 f^{(6)}[c]$$

These formulas can be written with the big "O" notation $O[h]^4$ and $O[h]^2$ if desired.

$$f'[x_0] = \frac{f[x_0-2h]-8f[x_0-h]+8f[x_0+h]-f[x_0+2h]}{12h}+O[h]^4$$

$$f''[x_0] = \frac{-f[x_0-2h]+16f[x_0-h]-30f[x_0]+16f[x_0+h]-f[x_0+2h]}{12h^2}+O[h]^4$$

$$f^{(3)}[x_0] = \frac{-f[x_0-2h]+2f[x_0-h]-2f[x_0+h]+f[x_0+2h]}{2h^3}+O[h]^2$$

$$f^{(4)}[x_0] = \frac{f[x_0-2h]-4f[x_0-h]+6f[x_0]-4f[x_0+h]+f[x_0+2h]}{h^4}+O[h]^2$$

Exploration

Explorations with the Higher Order Formulas

Backward difference formulas for numerical differentiation

Using the 4 points

$$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}\}$$

$$f'[x_0] \to \frac{11 f[x_0] - 2 f[-3h+x_0] + 9 f[-2h+x_0] - 18 f[-h+x_0]}{6h} + \frac{1}{4} h^3 f^{(4)}[c]$$

$$f''[x_0] \to \frac{2 f[x_0] - f[-3h+x_0] + 4 f[-2h+x_0] - 5 f[-h+x_0]}{h^2} + \frac{11}{12} h^2 f^{(4)}[c]$$

$$f^{(3)}[x_0] \to \frac{f[x_0] - f[-3h+x_0] + 3 f[-2h+x_0] - 3 f[-h+x_0]}{h^3} + \frac{3}{2} h f^{(4)}[c]$$

Forward difference formulas for numerical differentiation

Using the 4 points

$$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}\}$$

$$f'[x_0] \to \frac{-11 f[x_0] + 18 f[h+x_0] - 9 f[2h+x_0] + 2 f[3h+x_0]}{6h} - \frac{1}{4} h^3 f^{(4)}[c]$$

$$f''[x_0] \to \frac{2 f[x_0] - 5 f[h+x_0] + 4 f[2h+x_0] - f[3h+x_0]}{h^2} + \frac{11}{12} h^2 f^{(4)}[c]$$

$$f^{(3)}[x_0] \to \frac{-f[x_0] + 3 f[h+x_0] - 3 f[2h+x_0] + f[3h+x_0]}{h^3} - \frac{3}{2} h f^{(4)}[c]$$

Exploration

Exploration

Central difference formulas for numerical differentiation

Using the 5 points

$\{\{x_0, f[-2h+x_0]\}, \{x_0, f[-h+x_0]\}, \{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}\}$

$$f'[x_0] \to \frac{f[-2h+x_0]-8f[-h+x_0]+8f[h+x_0]-f[2h+x_0]}{12h} + \frac{1}{30} h^4 f^{(5)}[c]$$

$$f''[x_0] \to \frac{-30f[x_0]-f[-2h+x_0]+16f[-h+x_0]+16f[h+x_0]-f[2h+x_0]}{12h^2} + \frac{1}{90} h^4 f^{(6)}[c]$$

$$f^{(3)}[x_0] \to \frac{-f[-2h+x_0]+2f[-h+x_0]-2f[h+x_0]+f[2h+x_0]}{2h^3} - \frac{1}{4} h^2 f^{(5)}[c]$$

$$f^{(4)}[x_0] \to \frac{6f[x_0]+f[-2h+x_0]-4f[-h+x_0]-4f[h+x_0]+f[2h+x_0]}{h^4} - \frac{1}{6} h^2 f^{(6)}[c]$$

Central difference formulas for numerical differentiation

Using the 9 points

$\{\{x_0, f[-4h+x_0]\}, \{x_0, f[-3h+x_0]\}, \{x_0, f[-2h+x_0]\}, \{x_0, f[-h+x_0]\}, \{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}, \{x_0, f[4h+x_0]\}\}$

$$f'[x_0] \to \frac{3f[-4h+x_0]-32f[-3h+x_0]+168f[-2h+x_0]-672f[-h+x_0]+672f[h+x_0]-168f[2h+x_0]+32f[3h+x_0]-3f[4h+x_0]}{840h} + \frac{1}{630} h^8 f^{(9)}[c]$$

$$f''[x_0] \to \frac{-14350f[x_0]-9f[-4h+x_0]+128f[-3h+x_0]-1008f[-2h+x_0]+8064f[-h+x_0]+8064f[h+x_0]-1008f[2h+x_0]+128f[3h+x_0]-9f[4h+x_0]}{5040h^2} + \frac{h^8 f^{(10)}[c]}{3150}$$

$$f^{(3)}[x_0] \to \frac{-7f[-4h+x_0]+72f[-3h+x_0]-338f[-2h+x_0]+488f[-h+x_0]-488f[h+x_0]+338f[2h+x_0]-72f[3h+x_0]+7f[4h+x_0]}{240h^3} - \frac{41h^6 f^{(9)}[c]}{3024}$$

$$f^{(4)}[x_0] \to \frac{2730f[x_0]+7f[-4h+x_0]-96f[-3h+x_0]+676f[-2h+x_0]-1952f[-h+x_0]-1952f[h+x_0]+676f[2h+x_0]-96f[3h+x_0]+7f[4h+x_0]}{240h^4} - \frac{41h^6 f^{(10)}[c]}{7560}$$

$$f^{(5)}[x_0] \to \frac{f[-4h+x_0]-9f[-3h+x_0]+26f[-2h+x_0]-29f[-h+x_0]+29f[h+x_0]-26f[2h+x_0]+9f[3h+x_0]-f[4h+x_0]}{6h^5} + \frac{13}{144} h^4 f^{(9)}[c]$$

$$f^{(6)}[x_0] \to \frac{-150f[x_0]-f[-4h+x_0]+12f[-3h+x_0]-52f[-2h+x_0]+116f[-h+x_0]+116f[h+x_0]-52f[2h+x_0]+12f[3h+x_0]-f[4h+x_0]}{4h^6} + \frac{13}{240} h^4 f^{(10)}[c]$$

$$f^{(7)}[x_0] \to \frac{-f[-4h+x_0]+6f[-3h+x_0]-14f[-2h+x_0]+14f[-h+x_0]-14f[h+x_0]+14f[2h+x_0]-6f[3h+x_0]+f[4h+x_0]}{2h^7} - \frac{5}{12} h^2 f^{(9)}[c]$$

$$f^{(8)}[x_0] \to \frac{70f[x_0]+f[-4h+x_0]-8f[-3h+x_0]+28f[-2h+x_0]-56f[-h+x_0]-56f[h+x_0]+28f[2h+x_0]-8f[3h+x_0]+f[4h+x_0]}{h^8} - \frac{1}{3} h^2 f^{(10)}[c]$$

Exploration

Backward difference formulas for numerical differentiation

Using the 4 points

$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}\}$

$$f'[x_0] \to \frac{11 f[x_0]-2 f[-3h+x_0]+9 f[-2h+x_0]-18 f[-h+x_0]}{6h} + \frac{1}{4} h^3 f^{(4)}[c]$$

$$f''[x_0] \to \frac{2 f[x_0]-f[-3h+x_0]+4 f[-2h+x_0]-5 f[-h+x_0]}{h^2} + \frac{11}{12} h^2 f^{(4)}[c]$$

$$f^{(3)}[x_0] \to \frac{f[x_0]-f[-3h+x_0]+3 f[-2h+x_0]-3 f[-h+x_0]}{h^3} + \frac{3}{2} h f^{(4)}[c]$$

Forward difference formulas for numerical differentiation

Using the 4 points

$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}\}$

$$f'[x_0] \to \frac{-11 f[x_0]+18 f[h+x_0]-9 f[2h+x_0]+2 f[3h+x_0]}{6h} - \frac{1}{4} h^3 f^{(4)}[c]$$

$$f''[x_0] \to \frac{2 f[x_0]-5 f[h+x_0]+4 f[2h+x_0]-f[3h+x_0]}{h^2} + \frac{11}{12} h^2 f^{(4)}[c]$$

$$f^{(3)}[x_0] \to \frac{-f[x_0]+3 f[h+x_0]-3 f[2h+x_0]+f[3h+x_0]}{h^3} - \frac{3}{2} h f^{(4)}[c]$$

Backward difference formulas for numerical differentiation

Using the 5 points

$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}, \{x_0, f[4h+x_0]\}\}$

$$f'[x_0] \to \frac{25 f[x_0]+3 f[-4h+x_0]-16 f[-3h+x_0]+36 f[-2h+x_0]-48 f[-h+x_0]}{12h} + \frac{1}{5} h^4 f^{(5)}[c]$$

$$f''[x_0] \to \frac{35 f[x_0]+11 f[-4h+x_0]-56 f[-3h+x_0]+114 f[-2h+x_0]-104 f[-h+x_0]}{12h^2} + \frac{5}{6} h^3 f^{(5)}[c]$$

$$f^{(3)}[x_0] \to \frac{5 f[x_0]+3 f[-4h+x_0]-14 f[-3h+x_0]+24 f[-2h+x_0]-18 f[-h+x_0]}{2h^3} + \frac{7}{4} h^2 f^{(5)}[c]$$

$$f^{(4)}[x_0] \to \frac{f[x_0]+f[-4h+x_0]-4 f[-3h+x_0]+6 f[-2h+x_0]-4 f[-h+x_0]}{h^4} + 2 h f^{(5)}[c]$$

Forward difference formulas for numerical differentiation

Using the 5 points

$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}, \{x_0, f[4h+x_0]\}\}$

$$f'[x_0] \to \frac{-25f[x_0]+48f[h+x_0]-36f[2h+x_0]+16f[3h+x_0]-3f[4h+x_0]}{12h} + \frac{1}{5}h^4 f^{(5)}[c]$$

$$f''[x_0] \to \frac{35f[x_0]-104f[h+x_0]+114f[2h+x_0]-56f[3h+x_0]+11f[4h+x_0]}{12h^2} - \frac{5}{6}h^3 f^{(5)}[c]$$

$$f^{(3)}[x_0] \to \frac{-5f[x_0]+18f[h+x_0]-24f[2h+x_0]+14f[3h+x_0]-3f[4h+x_0]}{2h^3} + \frac{7}{4}h^2 f^{(5)}[c]$$

$$f^{(4)}[x_0] \to \frac{f[x_0]-4f[h+x_0]+6f[2h+x_0]-4f[3h+x_0]+f[4h+x_0]}{h^4} - 2h f^{(5)}[c]$$

Backward difference formulas for numerical differentiation

Using the 6 points

$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}, \{x_0, f[4h+x_0]\}, \{x_0, f[5h+x_0]\}\}$

$$f'[x_0] \to \frac{137f[x_0]-12f[-5h+x_0]+75f[-4h+x_0]-200f[-3h+x_0]+300f[-2h+x_0]-300f[-h+x_0]}{60h} + \frac{1}{6}h^5 f^{(6)}[c]$$

$$f''[x_0] \to \frac{45f[x_0]-10f[-5h+x_0]+61f[-4h+x_0]-156f[-3h+x_0]+214f[-2h+x_0]-154f[-h+x_0]}{12h^2} + \frac{137}{180}h^4 f^{(6)}[c]$$

$$f^{(3)}[x_0] \to \frac{17f[x_0]-7f[-5h+x_0]+41f[-4h+x_0]-98f[-3h+x_0]+118f[-2h+x_0]-71f[-h+x_0]}{4h^3} + \frac{15}{8}h^3 f^{(6)}[c]$$

$$f^{(4)}[x_0] \to \frac{3f[x_0]-2f[-5h+x_0]+11f[-4h+x_0]-24f[-3h+x_0]+26f[-2h+x_0]-14f[-h+x_0]}{h^4} + \frac{17}{6}h^2 f^{(6)}[c]$$

$$f^{(5)}[x_0] \to \frac{f[x_0]-f[-5h+x_0]+5f[-4h+x_0]-10f[-3h+x_0]+10f[-2h+x_0]-5f[-h+x_0]}{h^5} + \frac{5}{2}h f^{(6)}[c]$$

Forward difference formulas for numerical differentiation

Using the 6 points

$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}, \{x_0, f[4h+x_0]\}, \{x_0, f[5h+x_0]\}\}$

$$f'[x_0] \to \frac{-137f[x_0]+300f[h+x_0]-300f[2h+x_0]+200f[3h+x_0]-75f[4h+x_0]+12f[5h+x_0]}{60h} - \frac{1}{6}h^5 f^{(6)}[c]$$

$$f''[x_0] \to \frac{45f[x_0]-154f[h+x_0]+214f[2h+x_0]-156f[3h+x_0]+61f[4h+x_0]-10f[5h+x_0]}{12h^2} + \frac{137}{180}h^4 f^{(6)}[c]$$

$$f^{(3)}[x_0] \to \frac{-17f[x_0]+71f[h+x_0]-118f[2h+x_0]+98f[3h+x_0]-41f[4h+x_0]+7f[5h+x_0]}{4h^3} - \frac{15}{8}h^3 f^{(6)}[c]$$

$$f^{(4)}[x_0] \to \frac{3f[x_0]-14f[h+x_0]+26f[2h+x_0]-24f[3h+x_0]+11f[4h+x_0]-2f[5h+x_0]}{h^4} + \frac{17}{6}h^2 f^{(6)}[c]$$

$$f^{(5)}[x_0] \to \frac{-f[x_0]+5f[h+x_0]-10f[2h+x_0]+10f[3h+x_0]-5f[4h+x_0]+f[5h+x_0]}{h^5} - \frac{5}{2}h f^{(6)}[c]$$

Backward difference formulas for numerical differentiation

Using the 7 points

$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}, \{x_0, f[4h+x_0]\}, \{x_0, f[5h+x_0]\}, \{x_0, f[6h+x_0]\}\}$

$$f'[x_0] \to \frac{147 f[x_0]+10 f[-6h+x_0]-72 f[-5h+x_0]+225 f[-4h+x_0]-400 f[-3h+x_0]+450 f[-2h+x_0]-360 f[-h+x_0]}{60 h} + \frac{1}{7} h^6 f^{(7)}[c]$$

$$f''[x_0] \to \frac{812 f[x_0]+137 f[-6h+x_0]-972 f[-5h+x_0]+2970 f[-4h+x_0]-5080 f[-3h+x_0]+5265 f[-2h+x_0]-3132 f[-h+x_0]}{180 h^2} + \frac{7}{10} h^5 f^{(7)}[c]$$

$$f^{(3)}[x_0] \to \frac{49 f[x_0]+15 f[-6h+x_0]-104 f[-5h+x_0]+307 f[-4h+x_0]-496 f[-3h+x_0]+461 f[-2h+x_0]-232 f[-h+x_0]}{8 h^3} + \frac{29}{15} h^4 f^{(7)}[c]$$

$$f^{(4)}[x_0] \to \frac{35 f[x_0]+17 f[-6h+x_0]-114 f[-5h+x_0]+321 f[-4h+x_0]-484 f[-3h+x_0]+411 f[-2h+x_0]-186 f[-h+x_0]}{6 h^4} + \frac{7}{2} h^3 f^{(7)}[c]$$

$$f^{(5)}[x_0] \to \frac{7 f[x_0]+5 f[-6h+x_0]-32 f[-5h+x_0]+85 f[-4h+x_0]-120 f[-3h+x_0]+95 f[-2h+x_0]-40 f[-h+x_0]}{2 h^5} + \frac{25}{6} h^2 f^{(7)}[c]$$

$$f^{(6)}[x_0] \to \frac{f[x_0]+f[-6h+x_0]-6 f[-5h+x_0]+15 f[-4h+x_0]-20 f[-3h+x_0]+15 f[-2h+x_0]-6 f[-h+x_0]}{h^6} + 3 h f^{(7)}[c]$$

Forward difference formulas for numerical differentiation

Using the 8 points

$\{\{x_0, f[x_0]\}, \{x_0, f[h+x_0]\}, \{x_0, f[2h+x_0]\}, \{x_0, f[3h+x_0]\}, \{x_0, f[4h+x_0]\}, \{x_0, f[5h+x_0]\}, \{x_0, f[6h+x_0]\}, \{x_0, f[7h+x_0]\}\}$

$$f'[x_0] \to \frac{-1089 f[x_0]+2940 f[h+x_0]-4410 f[2h+x_0]+4900 f[3h+x_0]-3675 f[4h+x_0]+1764 f[5h+x_0]-490 f[6h+x_0]+60 f[7h+x_0]}{420 h} - \frac{1}{8} h^7 f^{(8)}[c]$$

$$f''[x_0] \to \frac{938 f[x_0]-4014 f[h+x_0]+7911 f[2h+x_0]-9490 f[3h+x_0]+7380 f[4h+x_0]-3618 f[5h+x_0]+1019 f[6h+x_0]-126 f[7h+x_0]}{180 h^2} + \frac{363}{560} h^6 f^{(8)}[c]$$

$$f^{(3)}[x_0] \to \frac{-967 f[x_0]+5104 f[h+x_0]-11787 f[2h+x_0]+15560 f[3h+x_0]-12725 f[4h+x_0]+6432 f[5h+x_0]-1849 f[6h+x_0]+232 f[7h+x_0]}{120 h^3} - \frac{469}{240} h^5 f^{(8)}[c]$$

$$f^{(4)}[x_0] \to \frac{56 f[x_0]-333 f[h+x_0]+852 f[2h+x_0]-1219 f[3h+x_0]+1056 f[4h+x_0]-555 f[5h+x_0]+164 f[6h+x_0]-21 f[7h+x_0]}{6 h^4} + \frac{967}{240} h^4 f^{(8)}[c]$$

$$f^{(5)}[x_0] \to \frac{-46 f[x_0]+295 f[h+x_0]-810 f[2h+x_0]+1235 f[3h+x_0]-1130 f[4h+x_0]+621 f[5h+x_0]-190 f[6h+x_0]+25 f[7h+x_0]}{6 h^5} - \frac{35}{6} h^3 f^{(8)}[c]$$

$$f^{(6)}[x_0] \to \frac{4 f[x_0]-27 f[h+x_0]+78 f[2h+x_0]-125 f[3h+x_0]+120 f[4h+x_0]-69 f[5h+x_0]+22 f[6h+x_0]-3 f[7h+x_0]}{h^6} + \frac{23}{4} h^2 f^{(8)}[c]$$

$$f^{(7)}[x_0] \to \frac{-f[x_0]+7 f[h+x_0]-21 f[2h+x_0]+35 f[3h+x_0]-35 f[4h+x_0]+21 f[5h+x_0]-7 f[6h+x_0]+f[7h+x_0]}{h^7} - \frac{7}{2} h f^{(8)}[c]$$