7. Curvature

Given the function $y = f[x]$, the formula for the curvature (and radius of curvature) is stated in all calculus textbooks

Definition (Curvature) $\kappa[x] = \dfrac{f''[x]}{(1 + (f'[x])^2)^{3/2}}$,

Definition (Radius of Curvature) $\rho[x] = \dfrac{(1 + (f'[x])^2)^{3/2}}{f''[x]}$.

Definition (Osculating Circle) At the point $\{x, f[x]\}$ on the curve $y = f[x]$, the osculating circle is tangent to the curve and has radius $r[x]$.

Example 1. Consider the parabola $y = f[x] = x^2$ and the point $\{0, f[0]\} = (0, 0)$ on the curve. Find the radius of curvature and the circle of curvature.
Solution 1.

Finding Curvature at Any Point

For the above example the circle of curvature was easy to locate because it's center lies on the y-axis. How do you locate the center if the point of tangency is not the origin? To begin, we need the concepts of tangent and normal vectors.

Tangent and Normal Vectors

Given the graph $y = f[x]$, a vector tangent to the curve at the point $(x, f(x))$ is $\vec{v} = \{1, f'(x)\}$. The unit tangent vector is $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$, which can be written as

Definition (Unit Tangent) $\vec{u} = \dfrac{1}{\sqrt{1 + (f'[x])^2}}\{1, f'[x]\}$

For vectors in $\mathbb{R}^2$, a corresponding perpendicular vector called the unit normal vector is given by

Lemma (Unit Normal) $\vec{n} = \dfrac{1}{\sqrt{1 + (f'[x])^2}}\{-f'[x], 1\}$.

Example 2. Consider the parabola $y = f[x] = x^2$ and the point $(1, f(1)) = (1, 1)$ on the curve.
Find the unit tangent and unit normal at point $(1, 1)$.
Solution 2.

Example 3. Consider the parabola $y = f[x] = x^2$ and the point $(1, f(1)) = (1, 1)$ on the curve.
Find the radius of curvature and the circle of curvature.
Solution 3.

A new construction of the Circle of Curvature

What determines a circle? A center and a radius. The formula for the radius of curvature is well established. What idea could we use to help understand the situation. We could use the fact that three points determine a circle and see where this leads.

Example 4. Consider the parabola $y = f[x] = x^2$ and the point $(0, f(0)) = (0, 0)$ on the curve.
Find collocation circle to go through the three points $(-h, f(-h))$, $(0, f(0))$, and $(h, f(h))$, and explore the situation for h = 1,.1,.01.
Solution 4.,

Derivation of the Radius of Curvature

The standard derivation of the formula for radius curvature involves the rate of change of the unit tangent vector. This new derivation starts with the collocation the collocation circle to go through the three points $(x-h, f(x-h))$, $(x, f(x))$, and $(x+h, f(x+h))$ on the curve $y = f(x)$. The limit as $h \to 0$ is the osculating circle to the curve $y = f(x)$ at the point $(x, f(x))$. The radius of curvature and formulas for the location of its center are simultaneously derived.

Start with the equation $(x-a)^2 + (y-b)^2 = r^2$, of a circle. Then write down three equations that force the collocation circle to go through the three points $(x-h, f(x-h))$, $(x, f(x))$, and $(x+h, f(x+h))$ on the curve $y = f(x)$.

$$(-a-h+x)^2 + (-b+f[-h+x])^2 == r^2$$
$$(-a+x)^2 + (-b+f[x])^2 == r^2$$
$$(-a+h+x)^2 + (-b+f[h+x])^2 == r^2$$

Solve the equations for a, b and r and extract the formula for the radius of the collocation circle. Since it depends on x and r we will store it as the function $r[x, h]$.

Solve the equations

$$(-a-h+x)^2 + (-b+f[-h+x])^2 == r^2$$
$$(-a+x)^2 + (-b+f[x])^2 == r^2$$
$$(-a+h+x)^2 + (-b+f[h+x])^2 == r^2$$

Get

$$r[x,h] = \frac{\sqrt{h^2 + (f[x]-f[-h+x])^2}\ \sqrt{h^2 + (f[x]-f[h+x])^2}\ \sqrt{4h^2 + (f[-h+x]-f[h+x])^2}}{2h(-2f[x] + f[-h+x] + f[h+x])}$$

The formula looks bewildering and one may wonder if it is of any value.

$$r[x] = \lim_{h\to 0} r[x,h]$$

$$r[x] = \lim_{h\to 0} \frac{\sqrt{h^2 + (f[x]-f[-h+x])^2}\ \sqrt{h^2 + (f[x]-f[h+x])^2}\ \sqrt{4h^2 + (f[-h+x]-f[h+x])^2}}{2h(-2f[x] + f[-h+x] + f[h+x])}$$

$$r[x] = \frac{(1 + f'[x]^2)^{3/2}}{f''[x]}$$

Therefore, the limit in the numerator is $(1 + F'[x]^2)^{3/2}$. The difference quotient in the denominator is recognized as the numerical approximation formula for the second derivative, hence the is $F''[x]$.

The Osculating Circle

We now show that the limit of the collocation circle as $h \to 0$ is the osculating circle.
Now we want to find the center (a, b) of the osculating circle.

The abscissa for the center of the collocation circle is

The abscisssa of the center of the collocation circle

$$a[x,h] = \frac{f[x]^2\,(f[-h+x] - f[h+x]) + h\,(h - 2x)\,f[h+x] + f[-h+x]^2\,f[h+x] - f[-h+x]\,(h^2 + 2hx + f[h+x]^2) + f[x]\,(4hx - f[-h+x]^2 + f[h+x]^2)}{h\,(4f[x] - 2\,(f[-h+x] + f[h+x]))}$$

Take the limit as $h \to 0$, to obtain the abscissa for the center of the circle of curvature.

The abscisssa of the center of the circle of curvature

$$a[x] = \lim_{h\to 0} a[x,h]$$

$$a[x] = \lim_{h\to 0} \frac{f[x]^2\,(f[-h+x] - f[h+x]) + h\,(h - 2x)\,f[h+x] + f[-h+x]^2\,f[h+x] - f[-h+x]\,(h^2 + 2hx + f[h+x]^2) + f[x]\,(4hx - f[-h+x]^2 + f[h+x]^2)}{h\,(4f[x] - 2\,(f[-h+x] + f[h+x]))}$$

$$a[x] = -\frac{f'[x] + f'[x]^3 - x\,f''[x]}{f''[x]}$$

$$a[x] = x - \frac{f'[x]}{f''[x]} - \frac{f'[x]^3}{f''[x]}$$

The numerators involve three difference quotients, all of which tend to $F'[x]$ when $h \to 0$, and the difference quotient in the denominators tends to $F''[x]$ when $h \to 0$.

The Abscissa the Easy Way

Subtract from x the radius of curvature times $\frac{f'[x]}{\sqrt{1 + f'[x]^2}}$.

$$r[x] = \frac{(1 + f'[x]^2)^{3/2}}{f''[x]}$$

The abscisssa of the center of the circle of curvature

$$a[x] = x - \frac{f'[x]}{\sqrt{1 + f'[x]^2}}\,r[x]$$

$$a[x] = x - \frac{f'[x]}{\sqrt{1 + f'[x]^2}}\,\frac{(1 + f'[x]^2)^{3/2}}{f''[x]}$$

$$a[x] = x - \frac{f'[x]\,(1 + f'[x]^2)}{f''[x]}$$

$$a[x] = x - \frac{f'[x]}{f''[x]} - \frac{f'[x]^3}{f''[x]}$$

The Ordinate for the Center of the Circle of Curvature

The ordinate for the center of the collocation circle is

The ordinate of the center of the collocation circle

$$b[x,h] = \frac{2h^2 - 2f[x]^2 + f[-h+x]^2 + f[h+x]^2}{2(-2f[x] + f[-h+x] + f[h+x])}$$

The ordinate of the center of the circle of curvature

$$b[x] = \lim_{h\to 0} b[x,h]$$

$$b[x] = \lim_{h\to 0} \frac{2h^2 - 2f[x]^2 + f[-h+x]^2 + f[h+x]^2}{2(-2f[x] + f[-h+x] + f[h+x])}$$

$$b[x] = f[x] + \frac{1}{f''[x]} + \frac{f'[x]^2}{f''[x]}$$

Thus we have established the formula $b[x] = F[x] + \frac{1}{F''[x]} + \frac{F'[x]^2}{F''[x]}$ for the ordinate of center of the circle of curvature.

The Ordinate of the Circle the Easy Way

Add to $f[x]$ the radius of curvature times $\frac{1}{\sqrt{1+f'[x]^2}}$

$$r[x] = \frac{(1+f'[x]^2)^{3/2}}{f''[x]}$$

The abscisssa of the center of the circle of curvature

$$b[x] = f[x] - \frac{f'[x]}{\sqrt{1+f'[x]^2}} r[x]$$

$$b[x] = f[x] - \frac{f'[x]}{\sqrt{1+f'[x]^2}} \frac{(1+f'[x]^2)^{3/2}}{f''[x]}$$

$$b[x] = f[x] - \frac{f'[x]\,(1+f'[x]^2)}{f''[x]}$$

$$b[x] = f[x] - \frac{f'[x]}{f''[x]} - \frac{f'[x]^3}{f''[x]}$$

The Osculating Circle

At the point $\vec{p}[x] = \{x, f[x]\}$ on the curve $y = f[x]$, the center and radius of the osculating circle are given by the limits calculated in the preceding discussion.

Given the curve

$y = f[x]$

At the point

$\vec{p}[x] = \{x, f[x]\}$

The center and radius of the osculating circle are

$$\vec{c}[x] = \left\{x - \frac{f'[x]}{f''[x]} - \frac{f'[x]^3}{f''[x]},\ f[x] + \frac{1}{f''[x]} + \frac{f'[x]^2}{f''[x]}\right\}$$

$$r[x] = \frac{(1 + f'[x]^2)^{3/2}}{f''[x]}$$

Example 5. Consider the parabola $y = f(x) = x^2$ at the point $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, \frac{1}{4}\right)$. Draw the circle of curvature for various values

$h = 1.0, 0.9, 0.8, 0.7, 0.5, 0.4, 0.3, 0.2, 0.1, 0.08, 0.06, 0.04, 0.03, 0.02, 0.01, 0.0075, 0.005, 0.0025, 0.001$

Solution 5.

Generalizations for 2D

In two dimensions, a curve can be expressed with the parametric equations $x = f[t]$ and $y = g[t]$. Similarly, the formulas for the radius of curvature and center of curvature can be derived using limits. At the point $\vec{p}[t] = \{f[t], g[t]\}$ the center and radius of the circle of convergence is

$$\vec{c}[t] = \left\{f[t] + \frac{g'[t]\,(f'[t]^2 + g'[t]^2)}{g'[t]\,f''[t] - f'[t]\,g''[t]},\ g[t] + \frac{f'[t]\,(f'[t]^2 + g'[t]^2)}{f'[t]\,g''[t] - g'[t]\,f''[t]}\right\}$$

$$r[t] = \frac{(f'[t]^2 + g'[t]^2)^{3/2}}{|\,f'[t]\,g''[t] - g'[t]\,f''[t]\,|}$$

Remark. The absolute value is necessary, otherwise the formula would only work for a curve that is positively oriented.

The Abscissa the Easy Way

Subtract from $f[x]$ the radius of curvature times $\frac{g'[x]}{\sqrt{f'[x]^2 + g'[x]^2}}$. The abscissa of the circle of curvature is

$$a[t] = f[t] - \frac{g'[t]}{\sqrt{f'[t]^2 + g'[t]^2}}\, r[t]$$

$$a[t] = f[t] - \frac{g'[t]\,(f'[t]^2 + g'[t]^2)}{f'[t]\,g''[t] - g'[t]\,f''[t]}$$

The Ordinate the Easy Way

Add to $g[x]$ the radius of curvature times $\frac{f'[x]}{\sqrt{f'[x]^2 + g'[x]^2}}$. The ordinate of the circle of curvature is

$$b[t] = g[t] + \frac{f'[t]}{\sqrt{f'[t]^2 + g'[t]^2}} r[t]$$

$$b[t] = g[t] + \frac{f'[t]\,(f'[t]^2 + g'[t]^2)}{f'[t]\,g''[t] - g'[t]\,f''[t]}$$

Example 6. Consider the cardioid $x = 2\cos(t) - \cos(2t)$, $y = 2\sin(t) - \sin(2t)$. Draw the circle of curvature at $t = \frac{\pi}{3}, \frac{2\pi}{3}, \pi$.

Solution 6.

Example 7. Consider the cardioid $x = 2\cos(-t) - \cos(-2t)$, $y = 2\sin(-t) - \sin(-2t)$. Draw the circle of curvature at $t = \frac{\pi}{3}$.

Solution 7.

Example 1. Consider the parabola $y = f[x] = x^2$ and the point $\{0, f[0]\} = (0,0)$ on the curve. Find the radius of curvature and the circle of curvature.

Solution 1.

$$f[x] = x^2$$
$$f'[x] = 2x$$
$$f''[x] = 2$$

$$\rho[x] = (1 + (f'[x])^2)^{3/2}/(f''[x])$$
$$\rho[x] = (1 + (2x)^2)^{3/2}/(2)$$
$$\rho[x] = (1 + 4x^2)^{3/2}/(2)$$
$$\rho[x] = \frac{1}{2}(1+4x^2)^{3/2}$$

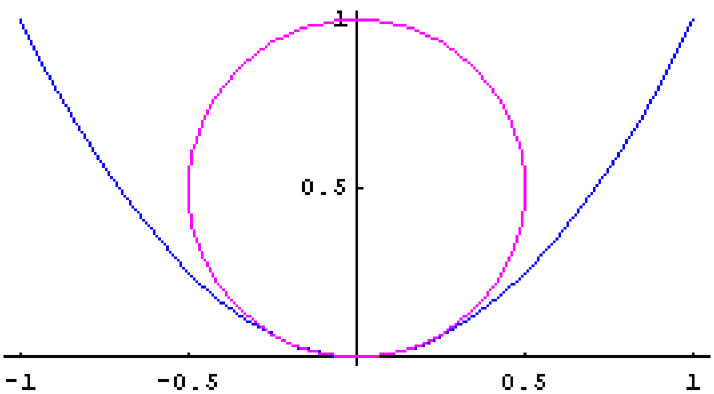
For $f[x] = x^2$

The radius of curvature at the point $\{x, f[x]\}$ is

$$\rho[x] = \frac{1}{2}(1+4x^2)^{3/2}$$

At $x=0$, we have $\rho[0] = \dfrac{(1 + (f'[0])^2)^{3/2}}{f''[0]} = \dfrac{1}{2}(1 + 4 * 0^2)^{3/2} = \dfrac{1}{2}$

We know the shape of this parabola and from symmetry we can conclude that the circle of curvature will have center $\left(0, \frac{1}{2}\right)$ and radius $\rho = \frac{1}{2}$.



$$y = f[x] = x^2$$

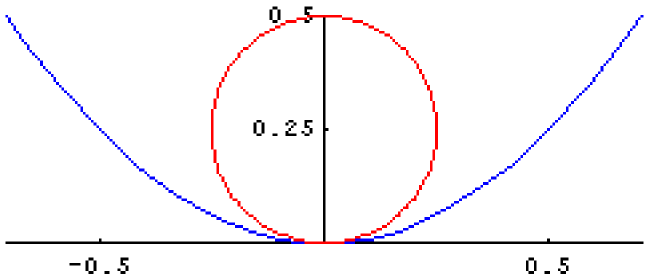
$$\rho[x] = \frac{1}{2}\left(1 + 4x^2\right)^{3/2}$$

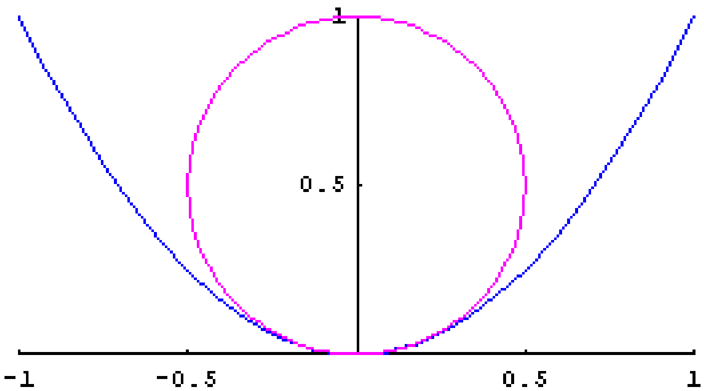
At the point

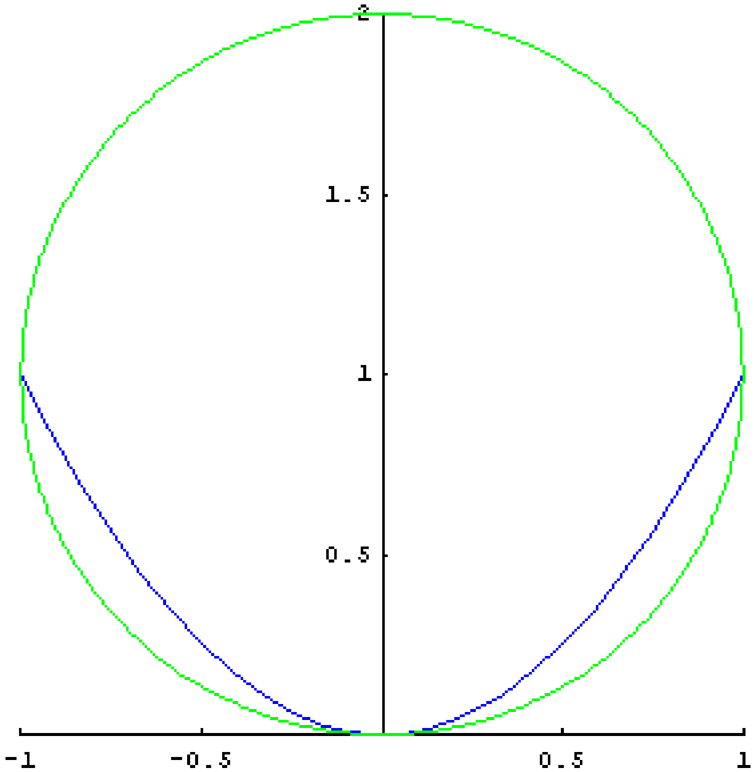
$$\{0, f[0]\} = \{0, 0\}$$

$$\rho[0] = \frac{1}{2}$$

Discussion. What do other kinds of "tangent circles" look like?







It looks like the "circle of curvature" is the "best fitting" circle.

Example 2. Consider the parabola $y = f[x] = x^2$ and the point $(1, f(1)) = (1, 1)$ on the curve. Find the unit tangent and unit normal at point $(1, 1)$.

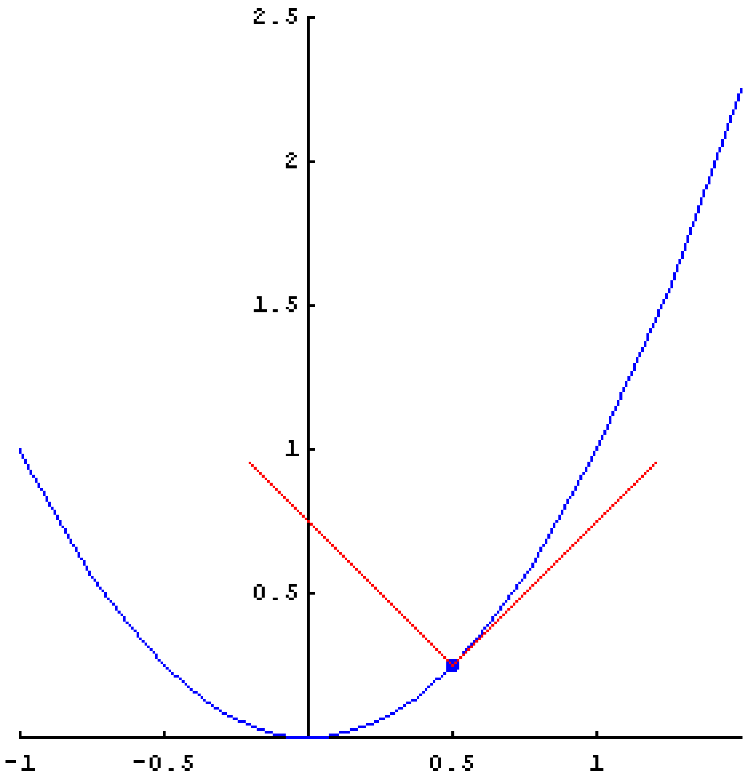
Solution 2.

$$y = f[x] = x^2$$

$$\vec{p}[x] = \{x, x^2\}$$

$$\vec{u}[x] = \left\{\frac{1}{\sqrt{1+4x^2}}, \frac{2x}{\sqrt{1+4x^2}}\right\}$$

$$\vec{n}[x] = \left\{-\frac{2x}{\sqrt{1+4x^2}}, \frac{1}{\sqrt{1+4x^2}}\right\}$$



$$y = f[x] = x^2$$

$$\vec{p}[\frac{1}{2}] = \left\{\frac{1}{2}, \frac{1}{4}\right\}$$

$$\vec{u}[\frac{1}{2}] = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

$$\vec{n}[\frac{1}{2}] = \left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

Example 3. Consider the parabola $y = f[x] = x^2$ and the point $(1,f(1)) = (1,1)$ on the curve. Find the radius of curvature and the circle of curvature.

Solution 3.

$y = f[x] = x^2$

$\vec{p}[\frac{1}{2}] = \{\frac{1}{2}, \frac{1}{4}\}$

$\rho[\frac{1}{2}] = \sqrt{2}$

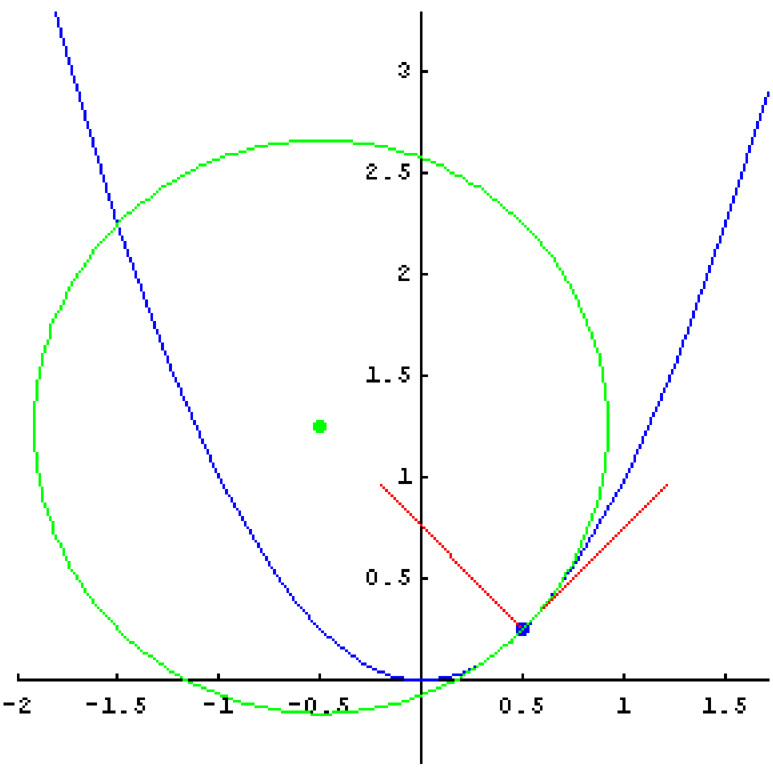
The center of the circle of convergence is

$\vec{c}[x] = \vec{p}[x] + \rho[x]\vec{n}[x]$

$\vec{c}[\frac{1}{2}] = \{\frac{1}{2}, \frac{1}{4}\} + \sqrt{2}\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$

$\vec{c}[\frac{1}{2}] = \{\frac{1}{2}, \frac{1}{4}\} + \{-1, 1\}$

$\vec{c}[\frac{1}{2}] = \{-\frac{1}{2}, \frac{5}{4}\}$



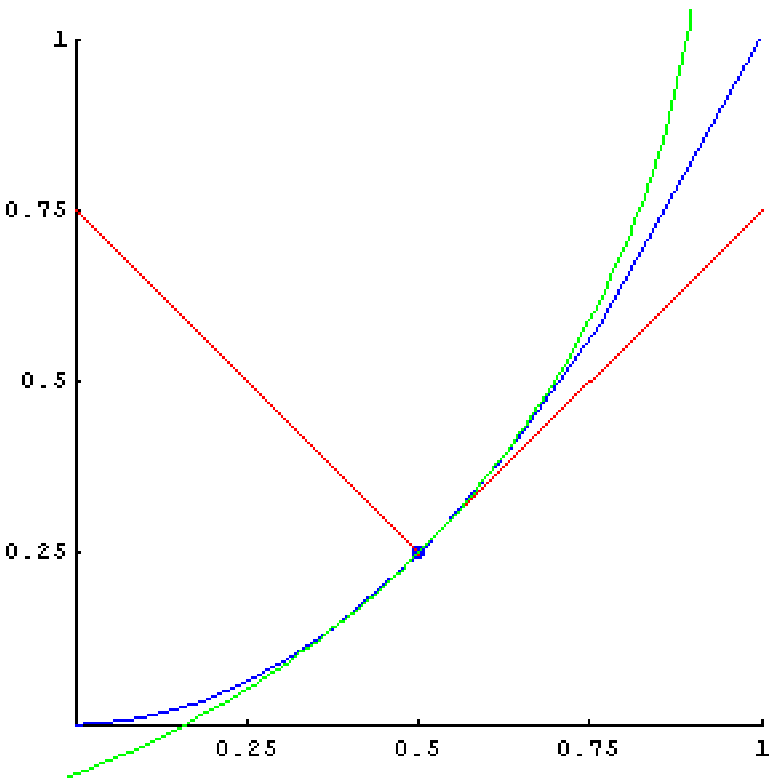
$y = f[x] = x^2$

$\vec{p}[\frac{1}{2}] = \{\frac{1}{2}, \frac{1}{4}\}$

The circle of convergence

$\rho[\frac{1}{2}] = \sqrt{2}$

$\vec{c}[\frac{1}{2}] = \{-\frac{1}{2}, \frac{5}{4}\}$



$y = f[x] = x^2$

$\vec{p}[\frac{1}{2}] = \{\frac{1}{2}, \frac{1}{4}\}$

The circle of convergence

$\rho[\frac{1}{2}] = \sqrt{2}$

$\vec{c}[\frac{1}{2}] = \{-\frac{1}{2}, \frac{5}{4}\}$

Example 4. Consider the parabola $y = f[x] = x^2$ and the point $(0,f(0)) = (0,0)$ on the curve. Find collocation circle to go through the three points $(-h, f(-h))$, $(0, f(0))$, and $(h, f(h))$, and explore the situation for h = 1,.1,.01.

Solution 4.

$y = f[x] = x^2$

Three points on the curve

$\{-h, f[-h]\} = \{-h, h^2\}$

$\{0, f[0]\} = \{0, 0\}$

$\{h, f[h]\} = \{h, h^2\}$

Start with the equation of a circle

$$(x - a)^2 + (y - b)^2 = r^2 .$$

Then write down three equations that force the collocation circle to go through the three points $(-h, f(-h))$, $(0, f(0))$, and $(h, f(h))$.

$$(-a - h)^2 + (-b + h^2)^2 == r^2$$
$$a^2 + b^2 == r^2$$
$$(-a + h)^2 + (-b + h^2)^2 == r^2$$

Expand the equations and get

$$a^2 + b^2 + 2 a h + h^2 - 2 b h^2 + h^4 == r^2$$
$$a^2 + b^2 == r^2$$
$$a^2 + b^2 - 2 a h + h^2 - 2 b h^2 + h^4 == r^2$$

Solve the equations for a, b and r and extract the formula for the radius of the collocation circle.

$$\{(-a - h)^2 + (-b + h^2)^2 == r^2, a^2 + b^2 == r^2, (-a + h)^2 + (-b + h^2)^2 == r^2\}$$

$$r \to \frac{1}{2}(-1 - h^2) \qquad a \to 0 \qquad b \to \frac{1}{2}(1 + h^2)$$
$$r \to \frac{1}{2}(1 + h^2) \qquad a \to 0 \qquad b \to \frac{1}{2}(1 + h^2)$$

$y = f[x] = x^2$ at the point $\{0,0\}$

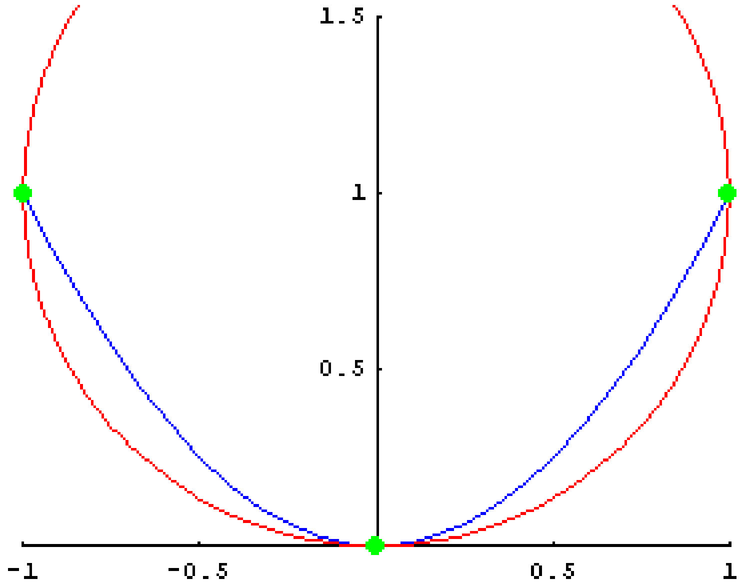
Radius of the collocation circle

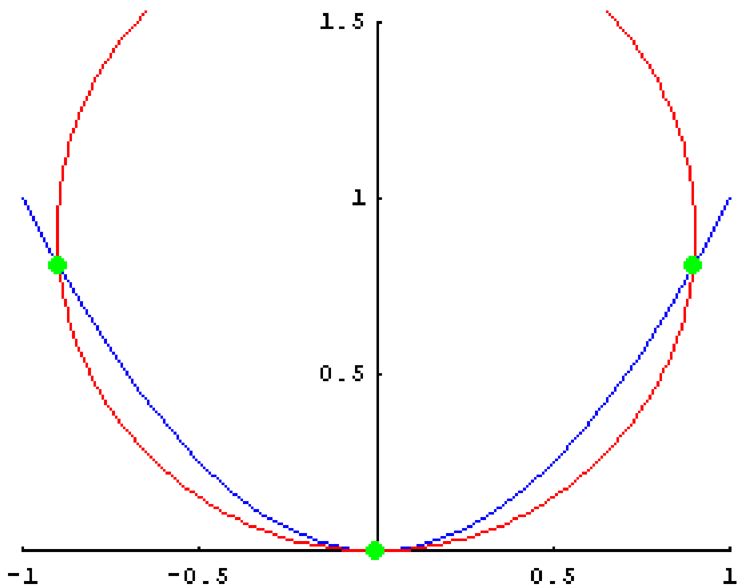
$$Rcc[h] = \frac{1}{2}(1 + h^2)$$

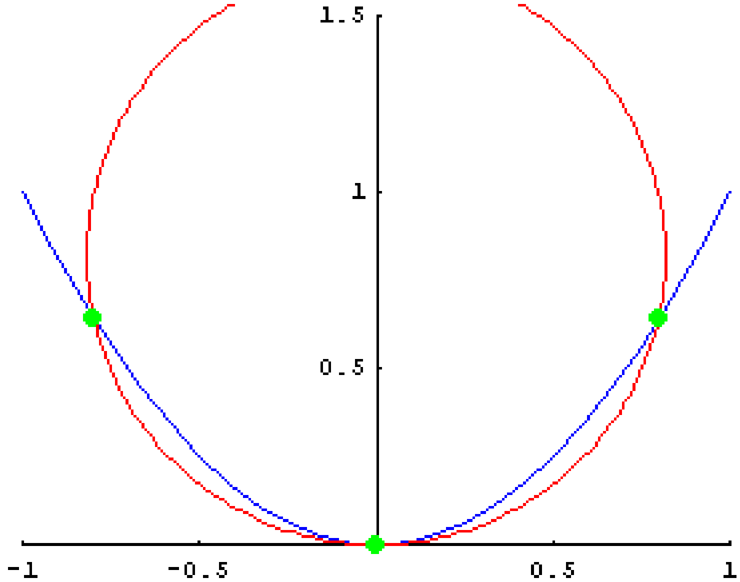
Animation.

Draw the circle of curvature for various values

h = 1.0, 0.9, 0.8, 0.7, 0.5, 0.4, 0.3, 0.2, 0.1, 0.08, 0.06, 0.04, 0.03, 0.02, 0.01



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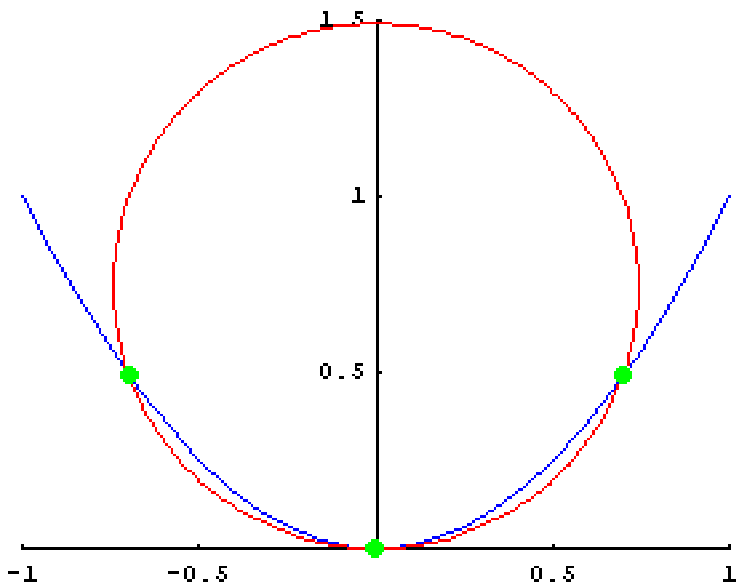
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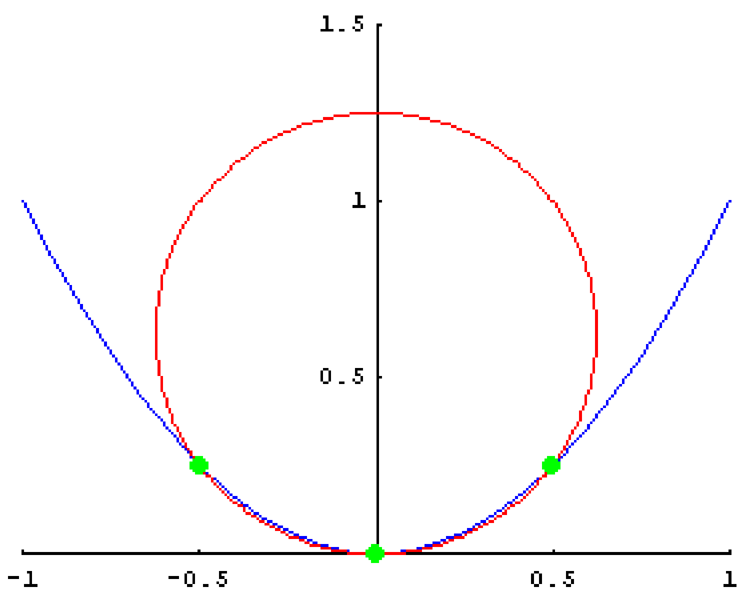
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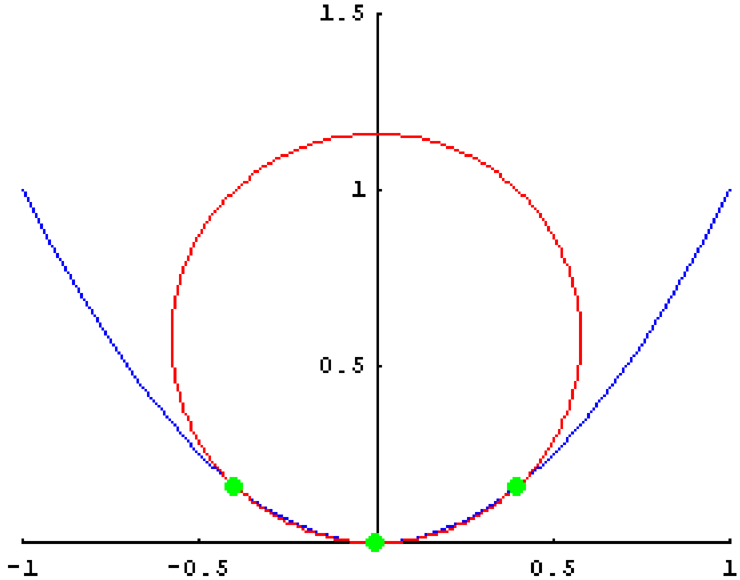
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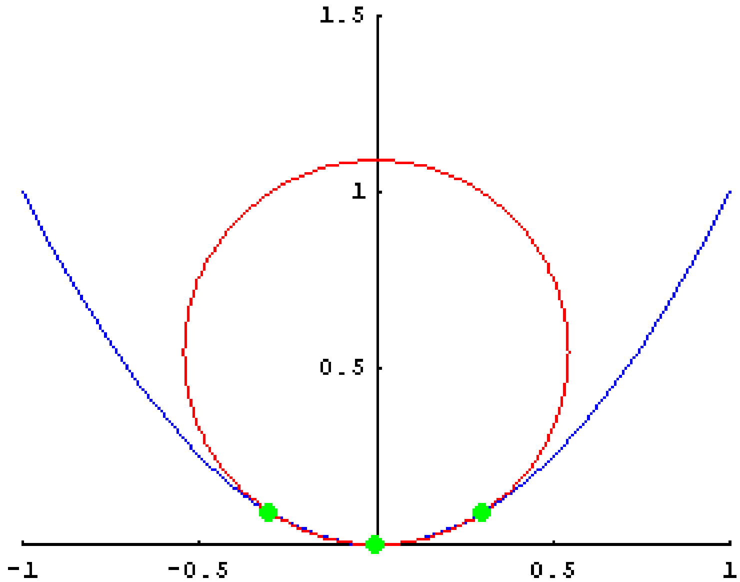
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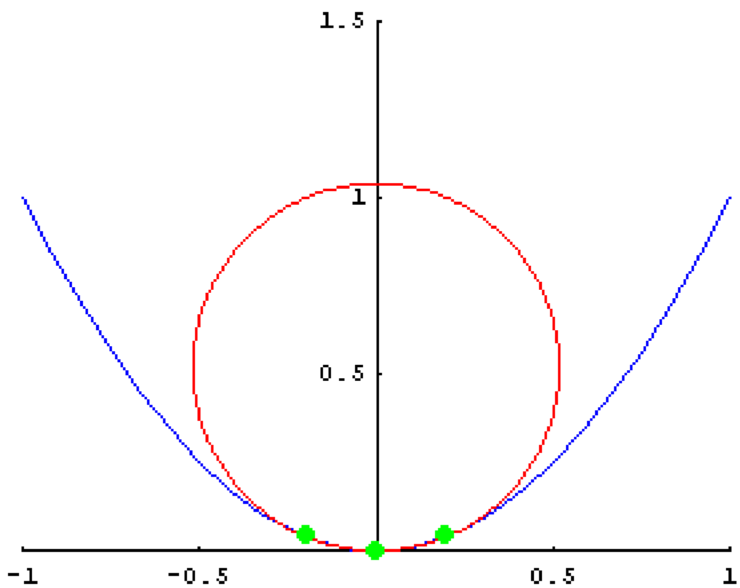
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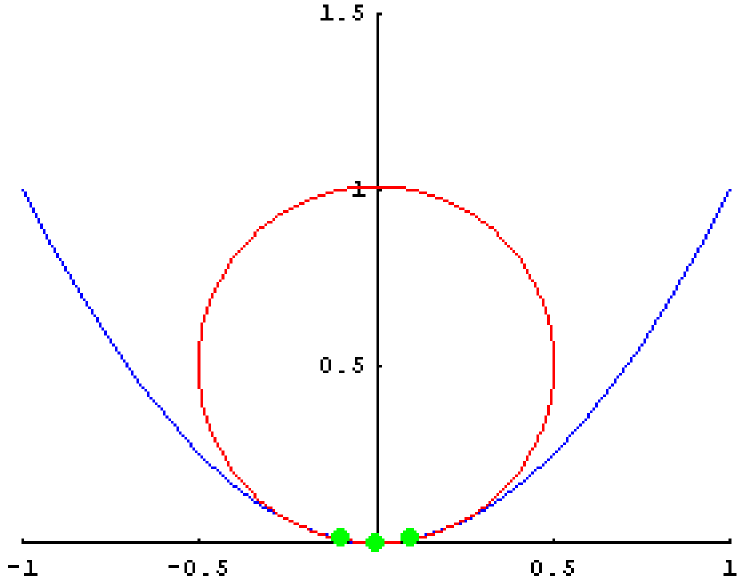
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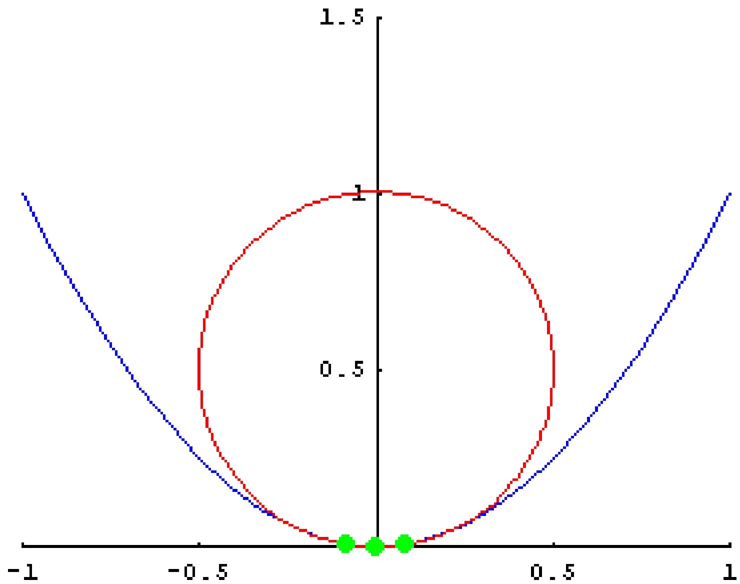
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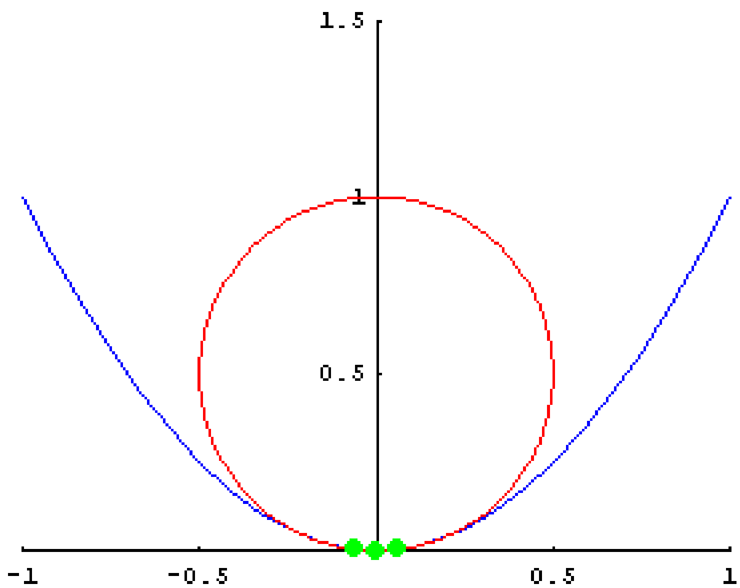
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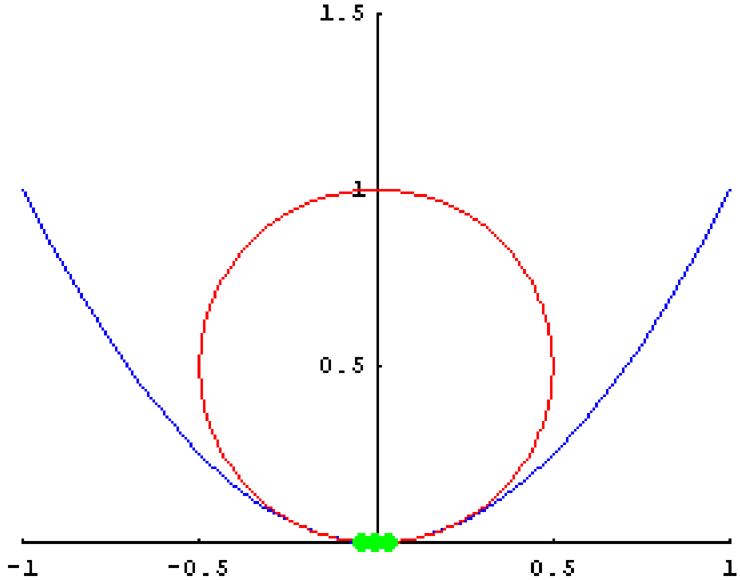
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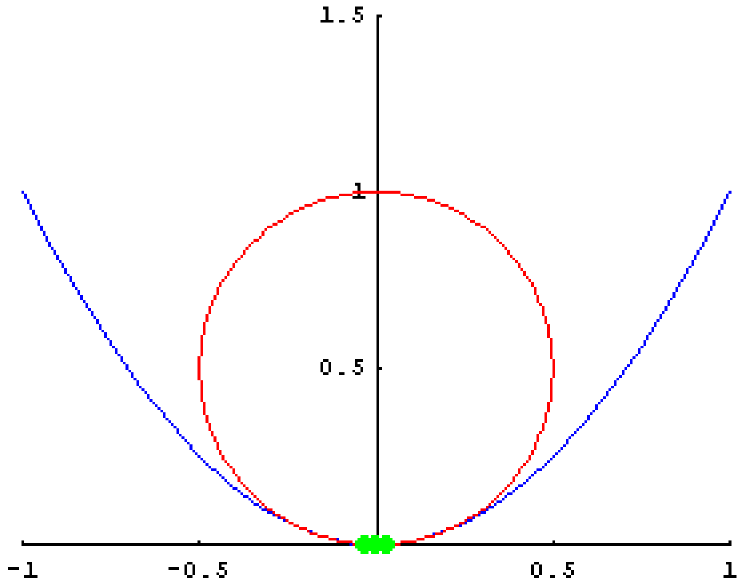
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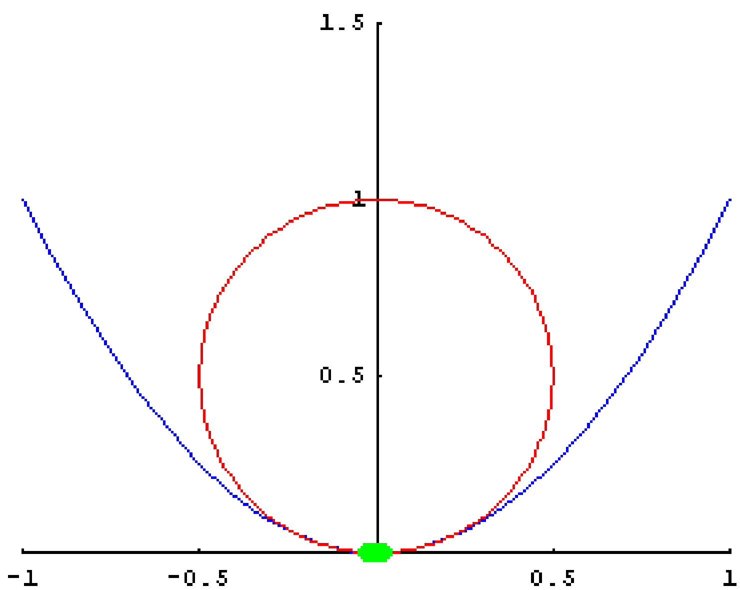
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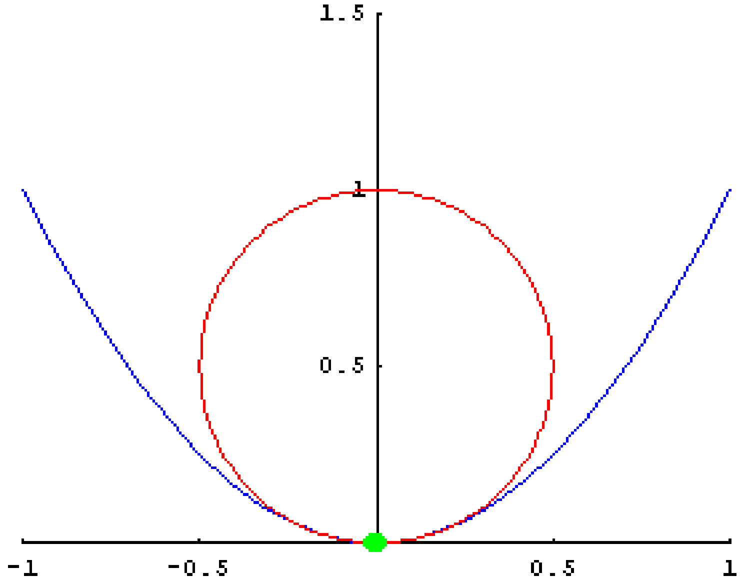


y = f[x] = x^2 at the point {0,0}

Radius of the collocation circle

h = 1. The radius = 1.

h = 0.9 The radius = 0.905

h = 0.8 The radius = 0.82

h = 0.7 The radius = 0.745

h = 0.5 The radius = 0.625

h = 0.4 The radius = 0.58

h = 0.3 The radius = 0.545

h = 0.2 The radius = 0.52

h = 0.1 The radius = 0.505

h = 0.08 The radius = 0.5032

h = 0.06 The radius = 0.5018

h = 0.04 The radius = 0.5008

h = 0.03 The radius = 0.50045

h = 0.02 The radius = 0.5002

h = 0.01 The radius = 0.50005

Take the limit and get the circle of curvature

The radius = 0.5

$y = f[x] = x^2$ at the point {0,0}

The radius of the collocation circle is

$$Rcc[h] = \frac{1}{2}(1 + h^2)$$

Take the limit as $h \to 0$

$$\lim_{h\to 0} Rcc[h] = \lim_{h\to 0} \frac{1}{2}(1 + h^2)$$

$$\lim_{h\to 0} Rcc[h] = \frac{1}{2}$$

It is the same as the radius of curvature at $x = 0$

$$\rho[x] = \frac{1}{2}(1 + 4x^2)^{3/2}$$

$$\rho[0] = \frac{1}{2}$$

We can conjecture that the limit of the collocation polynomial is the circle of curvature.

Example 5. Consider the parabola $y = f(x) = x^2$ at the point $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, \frac{1}{4}\right)$. Draw the circle of curvature for various values

$h = 1.0, 0.9, 0.8, 0.7, 0.5, 0.4, 0.3, 0.2, 0.1, 0.08, 0.06, 0.04, 0.03, 0.02, 0.01, 0.0075, 0.005, 0.0025, 0.001$

Solution 5.

Information for the animation

$$f[x] = x^2$$

$$\vec{p}[x] = \{x, x^2\}$$

The center and radius for the collocation circles at

$$\vec{p}[\tfrac{1}{2}] = \left\{\frac{1}{2}, \frac{1}{4}\right\}$$

$$a[\tfrac{1}{2},h] = \frac{\left(\frac{1}{2}-h\right)^4\left(\frac{1}{2}+h\right)^2 + (-1+h)\,h\left(\frac{1}{2}+h\right)^2 + \frac{1}{16}\left(\left(\frac{1}{2}-h\right)^2 - \left(\frac{1}{2}+h\right)^2\right) + \frac{1}{4}\left(-\left(\frac{1}{2}-h\right)^4 + 2h + \left(\frac{1}{2}+h\right)^4\right) - \left(\frac{1}{2}-h\right)^2\left(h + h^2 + \left(\frac{1}{2}+h\right)^4\right)}{h\left(1 - 2\left(\left(\frac{1}{2}-h\right)^2 + \left(\frac{1}{2}+h\right)^2\right)\right)}$$

$$b[\tfrac{1}{2},h] = \frac{-\frac{1}{8} + \left(\frac{1}{2}-h\right)^4 + 2h^2 + \left(\frac{1}{2}+h\right)^4}{2\left(-\frac{1}{2} + \left(\frac{1}{2}-h\right)^2 + \left(\frac{1}{2}+h\right)^2\right)}$$

$$r[\tfrac{1}{2},h] = \frac{\sqrt{\left(\frac{1}{4} - \left(\frac{1}{2}-h\right)^2\right)^2 + h^2}\,\sqrt{h^2 + \left(-\frac{1}{4} + \left(\frac{1}{2}+h\right)^2\right)^2}\,\sqrt{4h^2 + \left(-\left(\frac{1}{2}-h\right)^2 + \left(\frac{1}{2}+h\right)^2\right)^2}}{2h\left(-\frac{1}{2} + \left(\frac{1}{2}-h\right)^2 + \left(\frac{1}{2}+h\right)^2\right)}$$

At the point

$$\vec{p}[\tfrac{1}{2}] = \left\{\frac{1}{2}, \frac{1}{4}\right\}$$

Take the limit as $h \to 0$ and get the circle of curvature

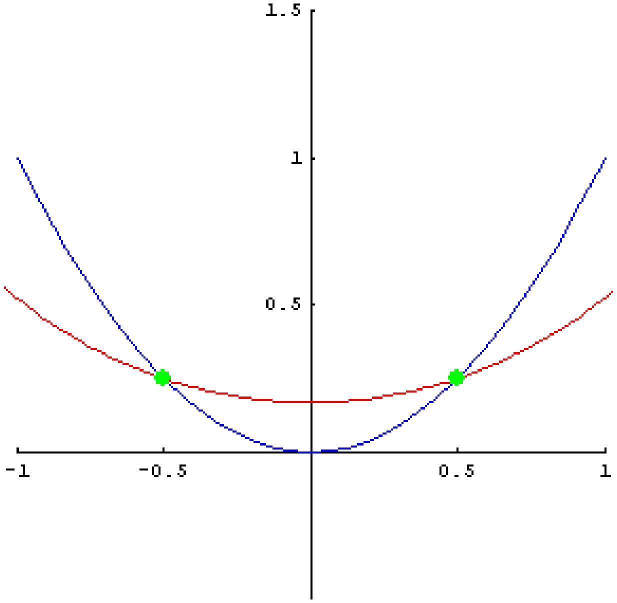
$$\lim_{h\to 0} a[\tfrac{1}{2},h] = -\frac{1}{2}$$

$$\lim_{h\to 0} b[\tfrac{1}{2},h] = \frac{5}{4}$$

$$\lim_{h\to 0} r[\tfrac{1}{2},h] = \sqrt{2}$$

$$\vec{c}[\tfrac{1}{2}] = \left\{-\frac{1}{2}, \frac{5}{4}\right\}$$

$$r[\tfrac{1}{2}] = \sqrt{2}$$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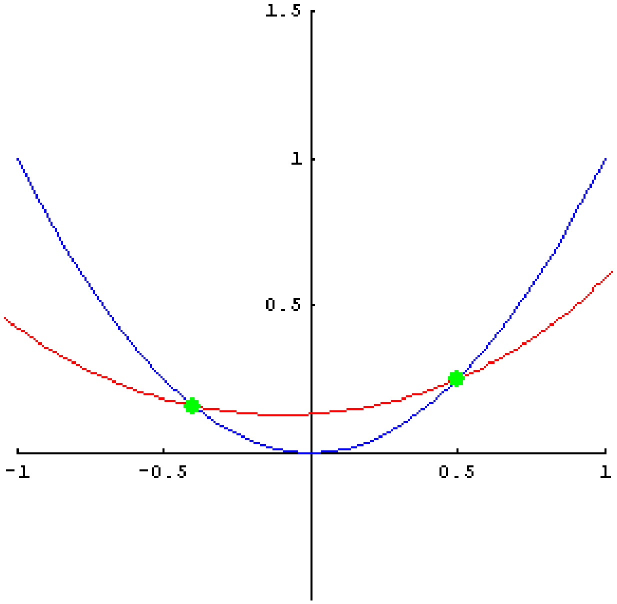
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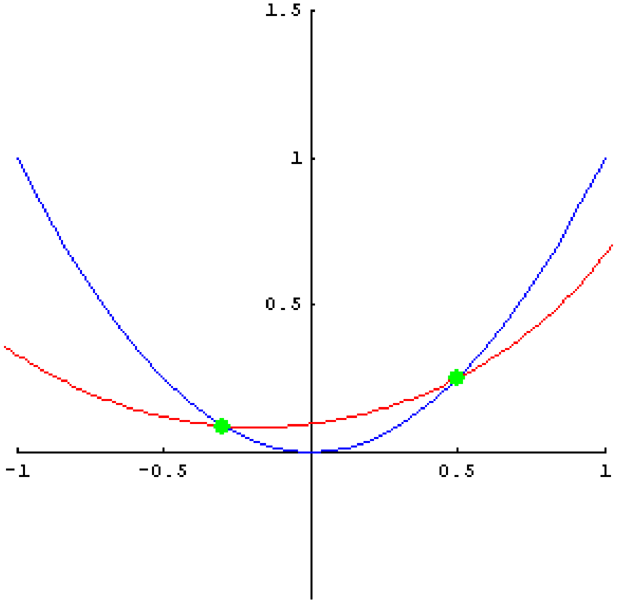
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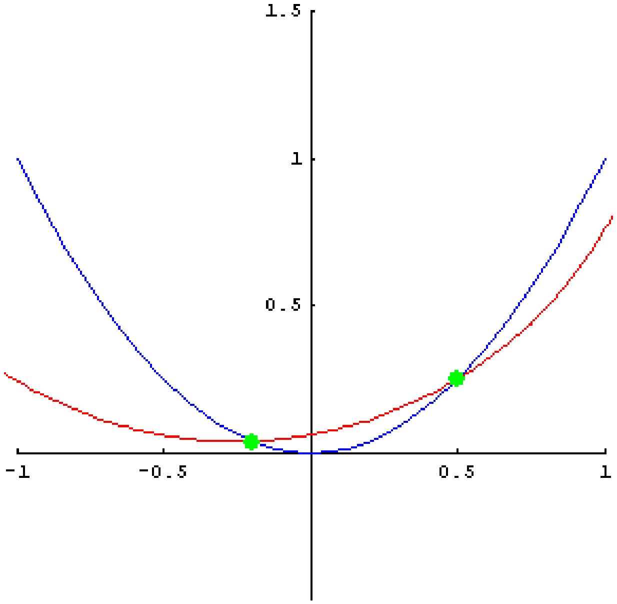
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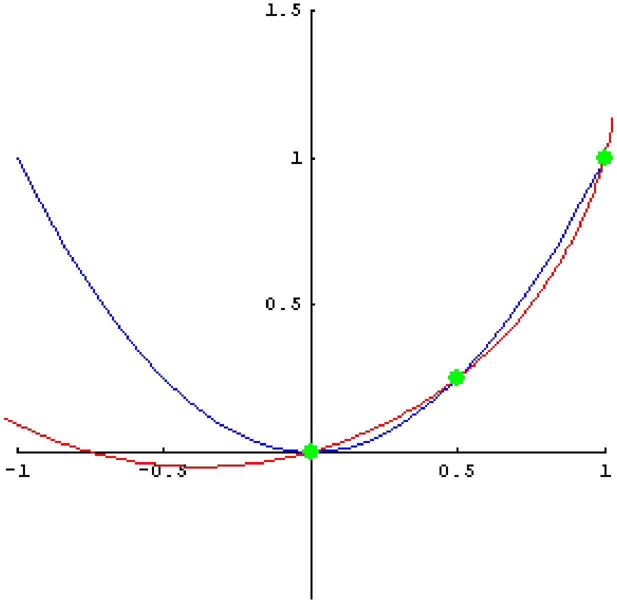
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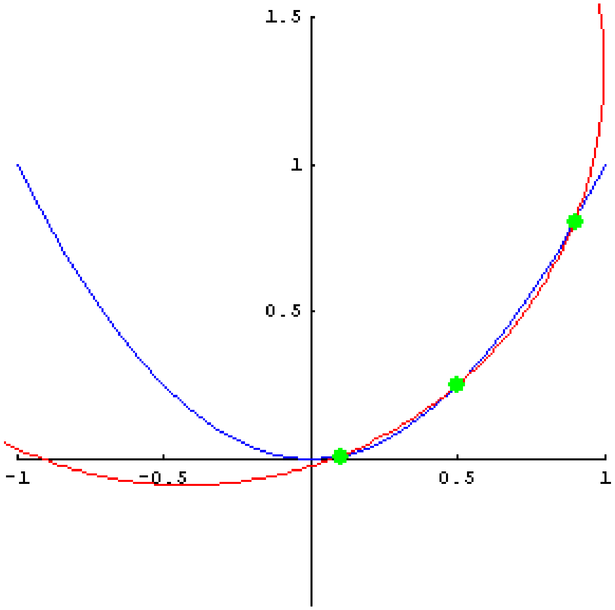
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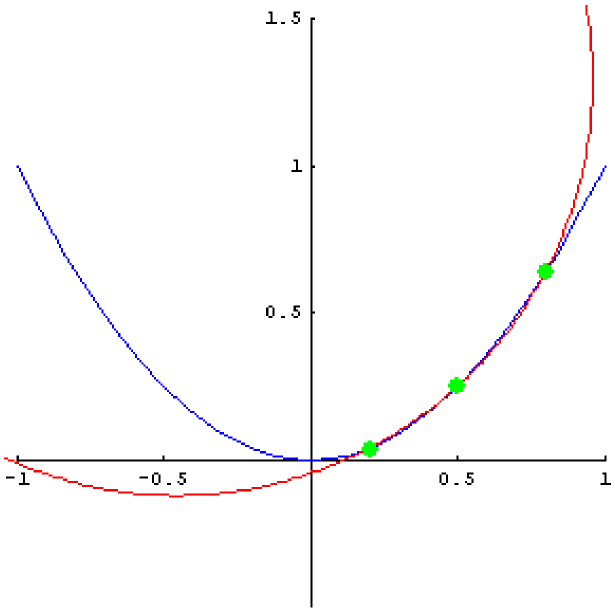
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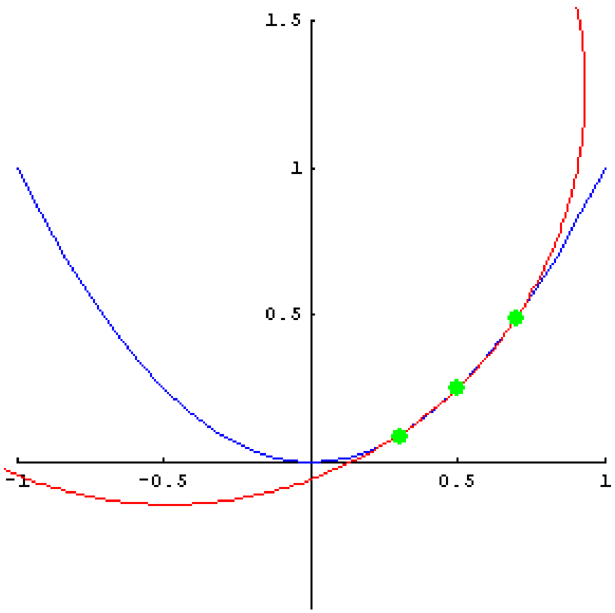
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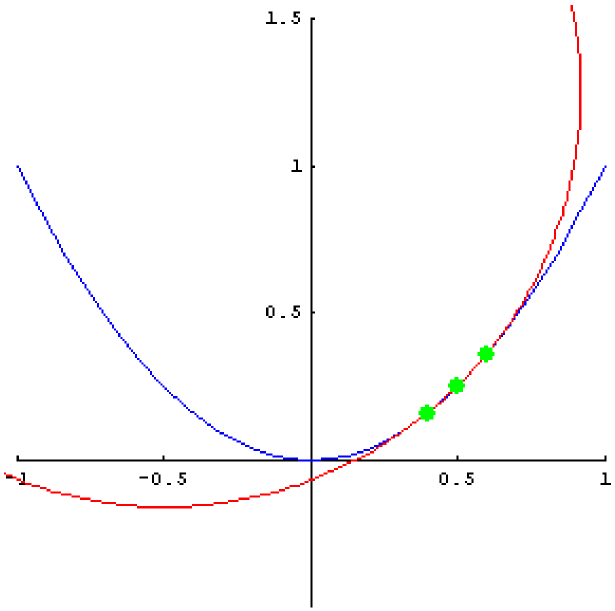
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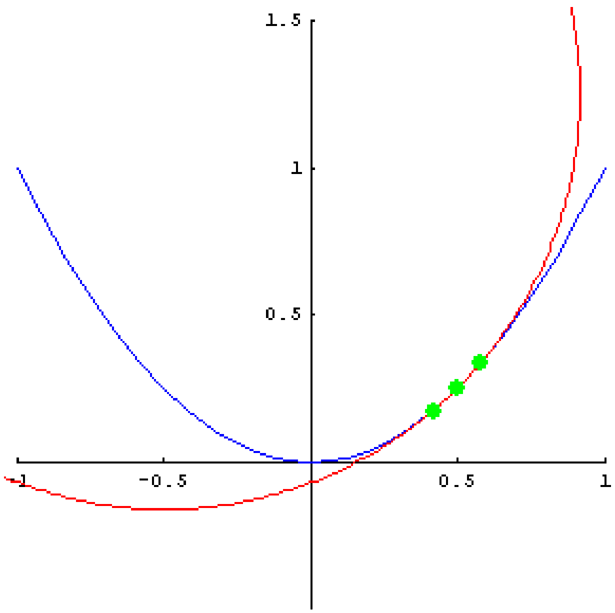
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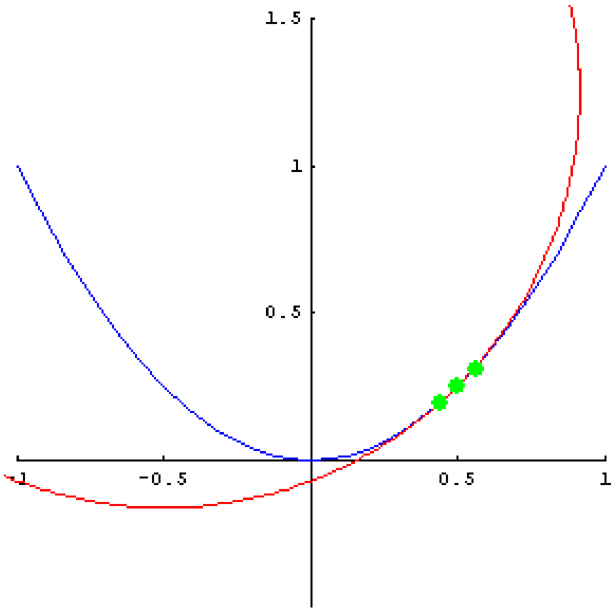
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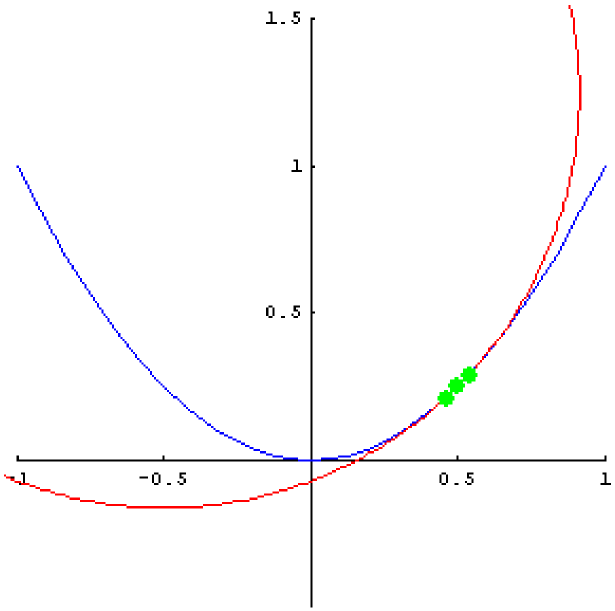
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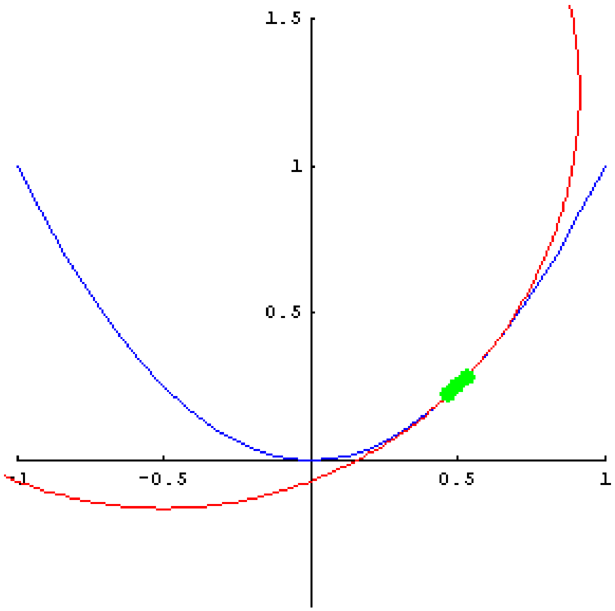
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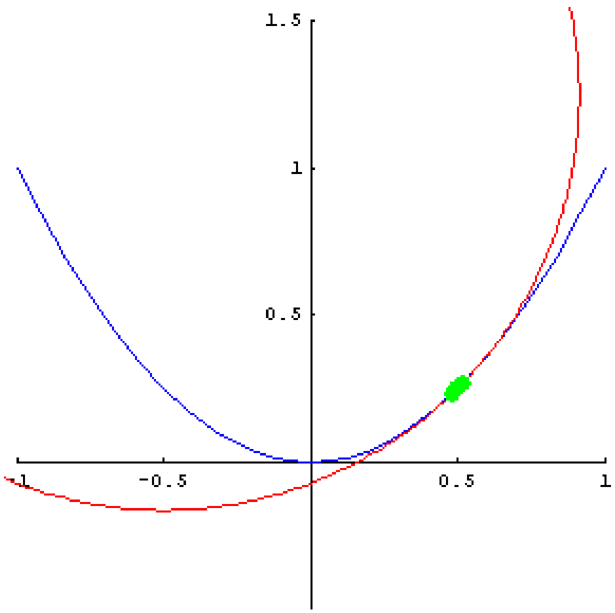
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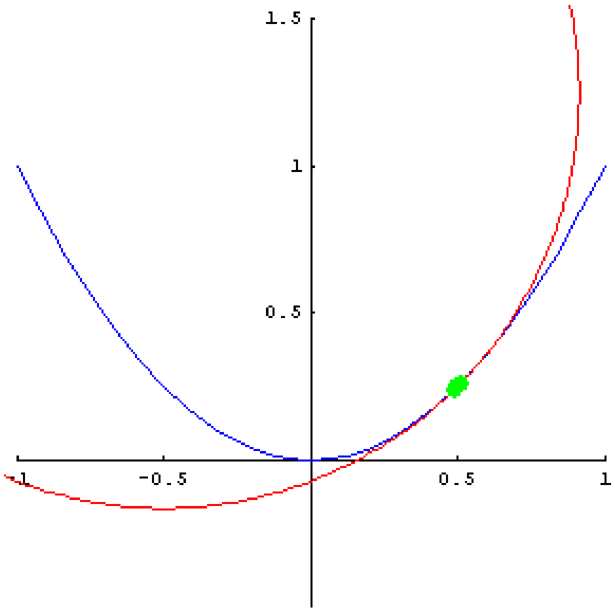
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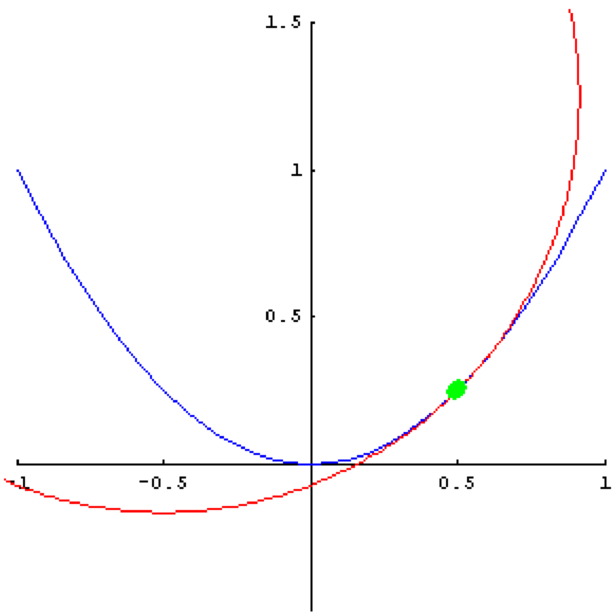
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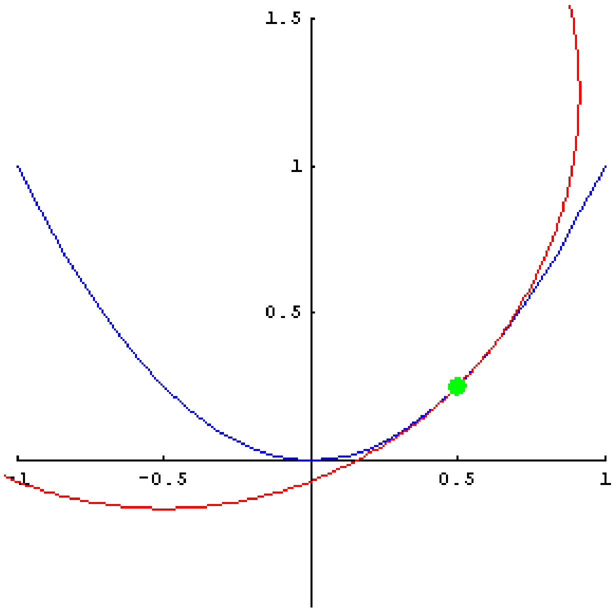
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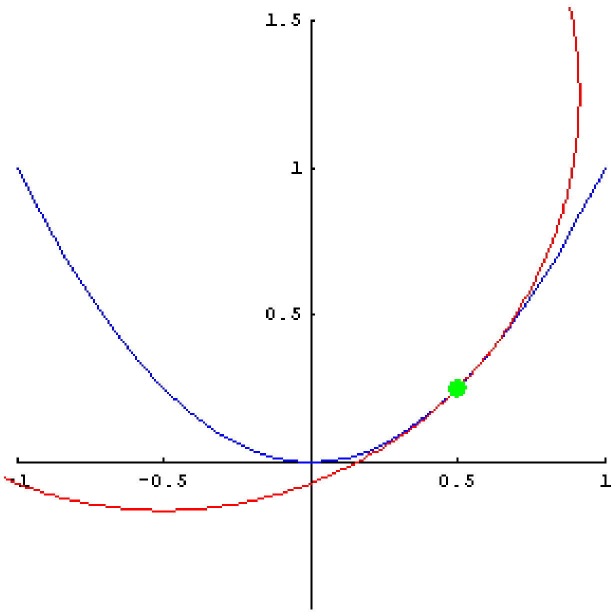
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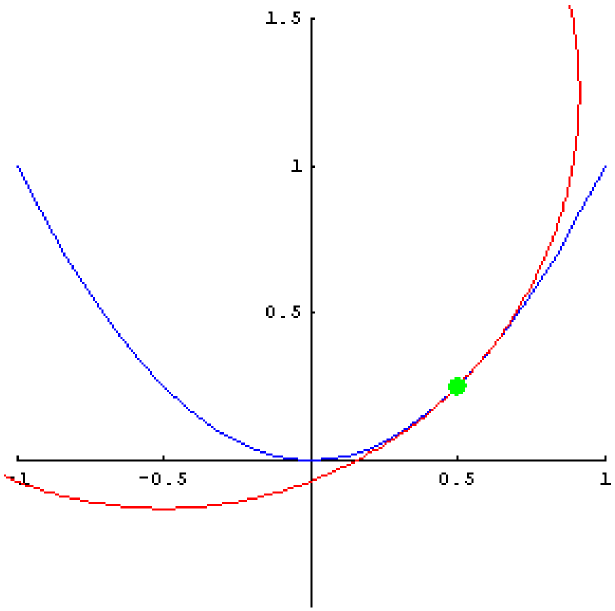
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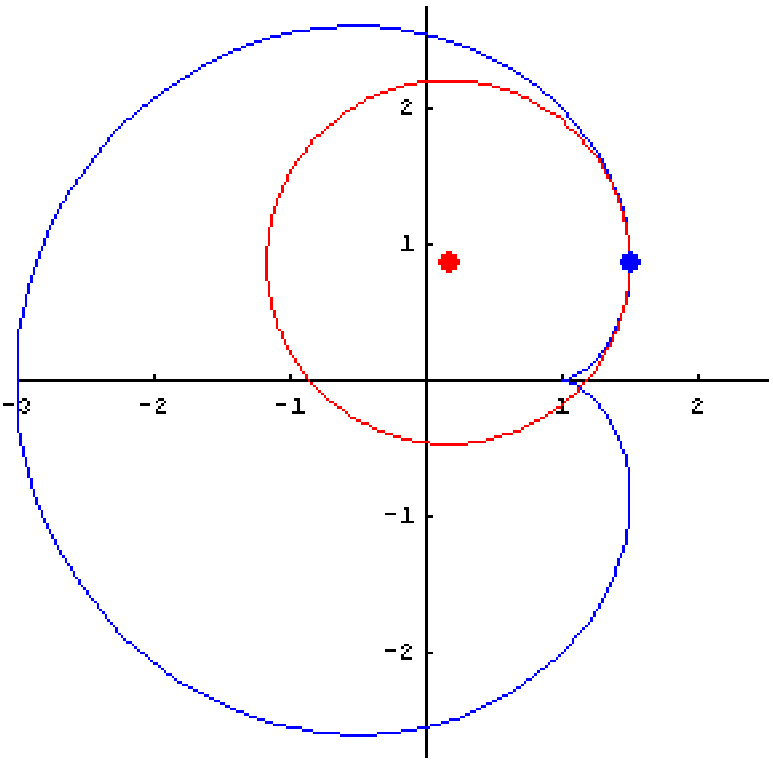
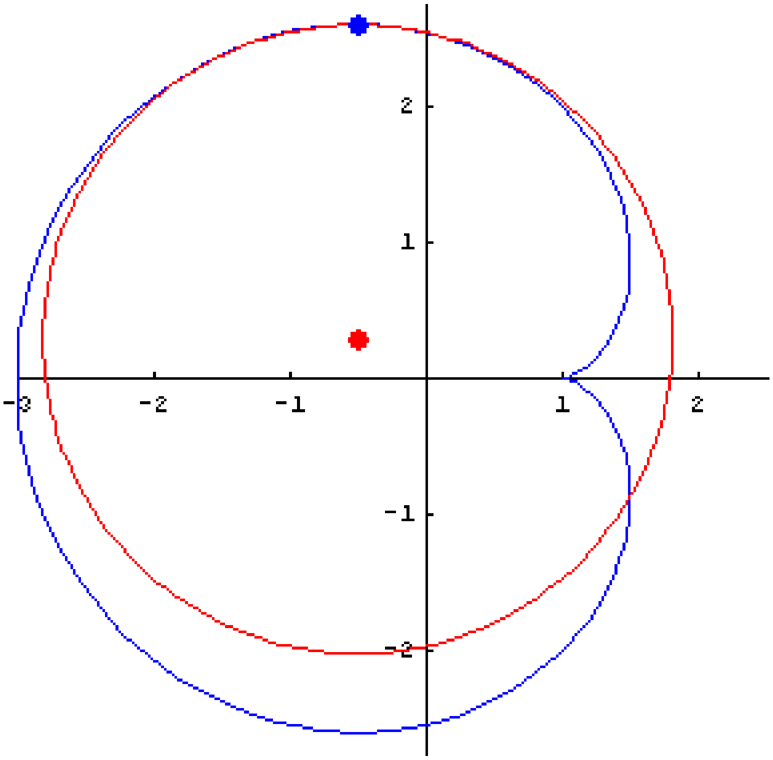
Example 6. Consider the cardioid $x = 2\cos(t) - \cos(2t)$, $y = 2\sin(t) - \sin(2t)$. Draw the circle of curvature at $t = \frac{\pi}{3}, \frac{2\pi}{3}, \pi$.

Solution 6.

$$x = 2\,\text{Cos}[t] - \text{Cos}[2\,t]$$
$$y = 2\,\text{Sin}[t] - \text{Sin}[2\,t]$$
$$\vec{p}[t] = \{2\,\text{Cos}[t] - \text{Cos}[2\,t],\ 2\,\text{Sin}[t] - \text{Sin}[2\,t]\}$$
$$\vec{c}[t] = \left\{\frac{1}{3}\,(2\,\text{Cos}[t] + \text{Cos}[2\,t]),\ \frac{2}{3}\,(1 + \text{Cos}[t])\,\text{Sin}[t]\right\}$$
$$r[t] = \frac{1}{24}\,(8 - 8\,\text{Cos}[t])^{3/2}\,\text{Csc}\left[\frac{t}{2}\right]^2$$



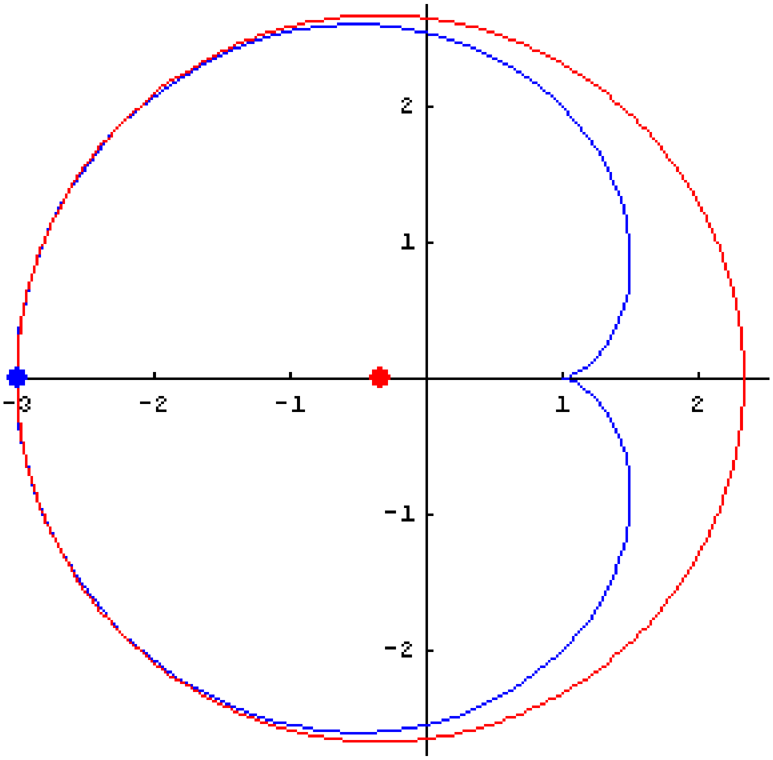
$$\vec{p}\left[\frac{\pi}{3}\right] = \left\{\frac{3}{2},\ \frac{\sqrt{3}}{2}\right\}$$
$$\vec{c}\left[\frac{\pi}{3}\right] = \left\{\frac{1}{6},\ \frac{\sqrt{3}}{2}\right\}$$
$$r\left[\frac{\pi}{3}\right] = \frac{4}{3}$$



$$\vec{p}\left[\frac{2\pi}{3}\right] = \left\{-\frac{1}{2}, \frac{3\sqrt{3}}{2}\right\}$$

$$\vec{c}\left[\frac{2\pi}{3}\right] = \left\{-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right\}$$

$$r\left[\frac{2\pi}{3}\right] = \frac{4}{\sqrt{3}}$$



$$\vec{p}[\pi] = \{-3, 0\}$$

$$\vec{c}[\pi] = \left\{-\frac{1}{3}, 0\right\}$$

$$r[\pi] = \frac{8}{3}$$

Example 7. Consider the cardioid $x = 2\cos(-t) - \cos(-2t)$, $y = 2\sin(-t) - \sin(-2t)$. Draw the circle of curvature at $t = \frac{\pi}{3}$.

Solution 7.

The radius of curvature formula $r[t] = \frac{(f'[t]^2 + g'[t]^2)^{3/2}}{f'[t]\,g''[t] - g'[t]\,f''[t]}$ can be used provided that the curve is positively oriented. Loosely speaking the curve must be oriented in the "counterclockwise direction". Let's investigate the situation at hand.

$$x = 2\,\text{Cos}[t] - \text{Cos}[2\,t]$$

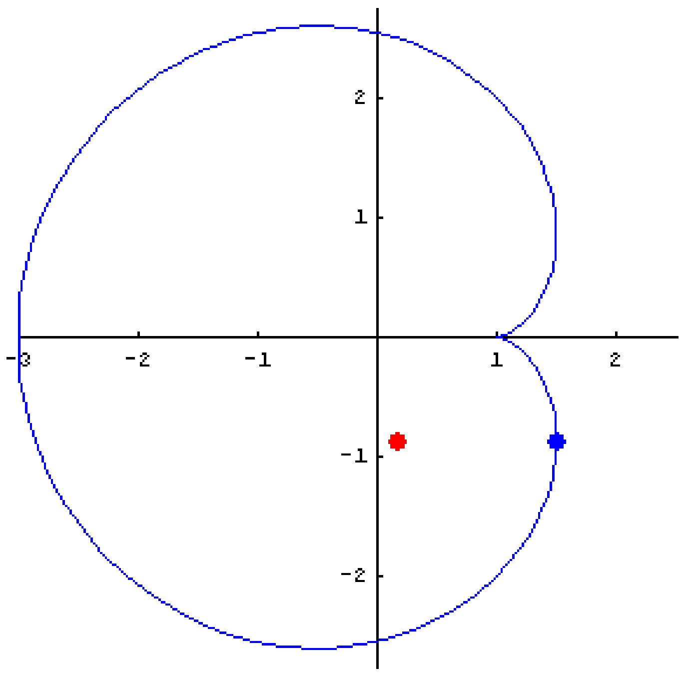
$$y = -2\,\text{Sin}[t] + \text{Sin}[2\,t]$$

$$\vec{p}[t] = \{2\,\text{Cos}[t] - \text{Cos}[2\,t],\ -2\,\text{Sin}[t] + \text{Sin}[2\,t]\}$$

$$\vec{c}[t] = \left\{\frac{1}{3}(2\,\text{Cos}[t] + \text{Cos}[2\,t]),\ -\frac{2}{3}(1 + \text{Cos}[t])\,\text{Sin}[t]\right\}$$

$$r[t] = -\frac{2}{3}\sqrt{8 - 8\,\text{Cos}[t]}$$

Circle::radius : Radius $-\frac{4}{3}$ is not a positive number or a pair of positive numbers.



$$\vec{p}\left[\frac{\pi}{3}\right] = \left\{\frac{3}{2},\ -\frac{\sqrt{3}}{2}\right\}$$

$$\vec{c}\left[\frac{\pi}{3}\right] = \left\{\frac{1}{6},\ -\frac{\sqrt{3}}{2}\right\}$$

$$r\left[\frac{\pi}{3}\right] = -\frac{4}{3}$$

The formulas failed because the curve is 'negatively oriented'

The formula for the radius of curvature computes a negative value. The correct the situation, change the sign and use

$$r[t] = -\frac{(f'[t]^2 + g'[t]^2)^{3/2}}{f'[t]\, g''[t] - g'[t]\, f''[t]} .$$

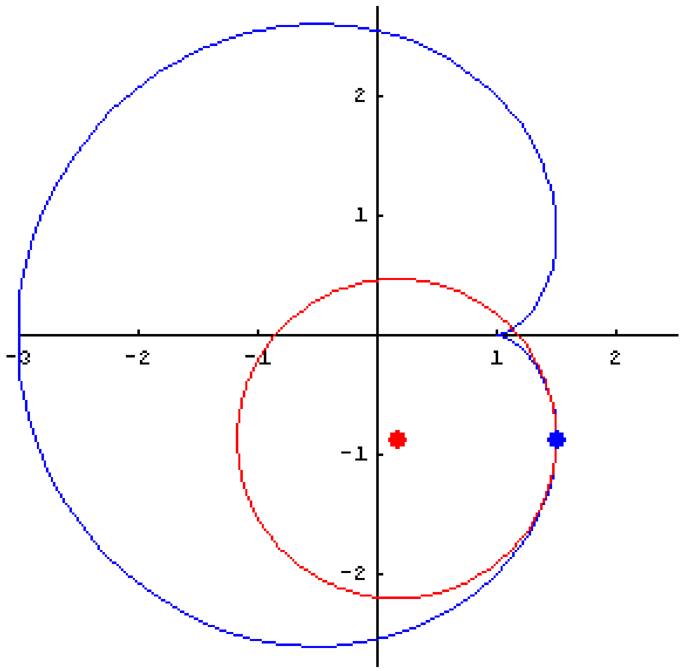
$$x = 2\,Cos[t] - Cos[2\,t]$$

$$y = -2\,Sin[t] + Sin[2\,t]$$

$$\vec{p}[t] = \{2\,Cos[t] - Cos[2\,t],\ -2\,Sin[t] + Sin[2\,t]\}$$

$$\vec{c}[t] = \left\{\frac{1}{3}\,(2\,Cos[t] + Cos[2\,t]),\ -\frac{2}{3}\,(1 + Cos[t])\,Sin[t]\right\}$$

$$r[t] = \frac{2}{3}\sqrt{8 - 8\,Cos[t]}$$



$$\vec{p}\left[\frac{\pi}{3}\right] = \left\{\frac{3}{2},\ -\frac{\sqrt{3}}{2}\right\}$$

$$\vec{c}\left[\frac{\pi}{3}\right] = \left\{\frac{1}{6},\ -\frac{\sqrt{3}}{2}\right\}$$

$$r\left[\frac{\pi}{3}\right] = \frac{4}{3}$$

Caveat. The formula for the radius of curvature should include an absolute value. Be careful! In *Mathematica* it could be written as

$$r[t] = Abs\left[\frac{(f'[t]^2 + g'[t]^2)^{3/2}}{f'[t]\, g''[t] - g'[t]\, f''[t]}\right].$$