5. The Secant Method

The Newton-Raphson algorithm requires two functions evaluations per iteration, $f(p_k)$ and $f'(p_k)$. Historically, the calculation of a derivative could involve considerable effort. But, with modern computer algebra software packages such as *Mathematica*, this has become less of an issue. Moreover, many functions have non-elementary forms (integrals, sums, discrete solution to an I.V.P.), and it is desirable to have a method for finding a root that does not depend on the computation of a derivative. The secant method does not need a formula for the derivative and it can be coded so that only one new function evaluation is required per iteration.

The formula for the secant method is the same one that was used in the regula falsi method, except that the logical decisions regarding how to define each succeeding term are different.

Theorem (Secant Method). Assume that $f \in C^2[a, b]$ and there exists a number $p \in [a, b]$, where $f(p) = 0$. If $f'(p) \neq 0$, then there exists a $\delta > 0$ such that the sequence $\{p_k\}_{k=0}^{\infty}$ defined by the iteration

$$p_{k+1} = g(p_{k-1}, p_k) = p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})} \quad \text{for} \quad k = 0, 1, \ldots$$

will converge to p for two initial approximations $p_0, p_1 \in [p - \delta, p + \delta]$.

Example 1. Use the secant method to find the three roots of the cubic polynomial $f[x] = 4x^3 - 16x^2 + 17x - 4$.
Determine the secant iteration formula $g[x] = x - \frac{f[x]}{f'[x]}$ that is used.
Show details of the computations for the starting value $p_0 = 3$ and $p_1 = 2.8$.
Solution 1.

Example 2. Use Newton's method to find the roots of the cubic polynomial $f[x] = x^3 - 3x + 2$.
2 (a) Fast Convergence. Investigate quadratic convergence at the simple root $p = -2$, using the starting value $p_0 = -2.6$ and $p_1 = -2.4$
2 (b) Slow Convergence. Investigate linear convergence at the double root $p = 1$, using the starting value $p_0 = 1.4$ and $p_1 = 1.2$
Solution 2.

Example 3. Fast Convergence Find the solution to $3\,\mathrm{Exp}[x] - 4\,\mathrm{Cos}[x] = 0$.
Use the Secant Method and the starting approximations $p_0 = 1.0$ and $p_1 = 0.9$.
Solution 3.

Example 4. NON Convergence, Diverging to Infinity Find the solution to $x\,e^{-x} = 0$.
Use the Secant Method and the starting approximations $p_0 = 1.5$ and $p_1 = 2.0$.

Solution 4.

Example 1. Use the secant method to find the three roots of the cubic polynomial $f[x] = 4x^3 - 16x^2 + 17x - 4$.

Determine the secant iteration formula $g[x] = x - \frac{f[x]}{f'[x]}$ that is used.

Show details of the computations for the starting value $p_0 = 3$ and $p_1 = 2.8$.

Solution 1.

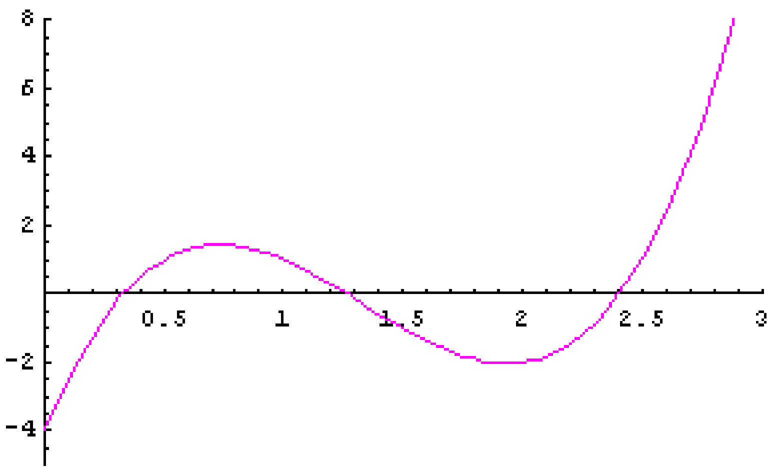
The secant iteration formula $g[x_0, x_1]$ is

$$p_2 = g[p_0, p_1] = p_1 - \frac{(-p_0 + p_1)(-4 + 17p_1 - 16p_1^2 + 4p_1^3)}{-17p_0 + 16p_0^2 - 4p_0^3 + 17p_1 - 16p_1^2 + 4p_1^3}$$

$$p_2 = g[p_0, p_1] = \frac{4(1 + p_0^2 p_1 + p_0(-4 + p_1)p_1)}{17 + 4p_0^2 + 4p_0(-4 + p_1) - 16p_1 + 4p_1^2}$$

Hopefully, the iteration $p_{n+1} = g[p_{n-1}, p_n]$ will converge to a root of $f[x]$.

Graph the function $y = f[x]$.



$$f[x] = -4 + 17x - 16x^2 + 4x^3$$

There are three real root.

Starting with the values $p_0 = 3$ and $p_1 = 2.8$.

Use the secant method to find a numerical approximation to the root.

First, do the iteration one step at a time.

$$p_0 = 3.0000000000000000, \quad f[p_0] = 11.$$

p_1 = 2.8000000000000000, $f[p_1]$ = 5.967999999999989
p_2 = 2.5627980922098570, $f[p_2]$ = 1.809778114233566
p_3 = 2.4595609795289590, $f[p_3]$ = 0.5373616335744913
p_4 = 2.4159623116721990, $f[p_4]$ = 0.0880463943210401
p_5 = 2.4074188545030930, $f[p_5]$ = 0.00584986163346457
p_6 = 2.4068108234745750, $f[p_6]$ = 0.0000718961762657954 8
p_7 = 2.4068032576442970, $f[p_7]$ = $6.000787777793448 \times 10^{-8}$
p_8 = 2.4068032513242300, $f[p_8]$ = $6.110667527536862 \times 10^{-13}$
p_9 = 2.4068032513241660, $f[p_9]$ = 0.
p_{10} = 2.4068032513241660, $f[p_{10}]$ = 0.
p = 2.406803251324166 Δp = ±0.
$f[p]$ = 0.

From the graph we see that there are two other real roots.
Use the starting values $p_0 = 0.6$ and $p_1 = 0.5$.

p_0 = 0.6000000000000000, $f[p_0]$ = 1.303999999999999
p_1 = 0.5000000000000000, $f[p_1]$ = 1.
p_2 = 0.1710526315789467, $f[p_2]$ = -1.540229989794438
p_3 = 0.3705048874540401, $f[p_3]$ = 0.3056437021120457
p_4 = 0.3374791576072164, $f[p_4]$ = 0.06861572052792555
p_5 = 0.3279187494467301, $f[p_5]$ = -0.004827220357223688
p_6 = 0.3285471311038295, $f[p_6]$ = 0.0000674885986464979 2
p_7 = 0.3285384669324079, $f[p_7]$ = $6.475412978046435 \times 10^{-8}$
p_8 = 0.3285384586113031, $f[p_8]$ = $-8.70692407062279 \times 10^{-13}$
p_9 = 0.3285384586114150, $f[p_9]$ = $1.387778780781446 \times 10^{-16}$
p_{10} = 0.3285384586114150, $f[p_{10}]$ = $1.387778780781446 \times 10^{-16}$

p = 0.328538458611415
Δp = ±0.
$f[p]$ = $1.387778780781446 \times 10^{-16}$

Use the starting values $p_0 = 1.0$ and $p_1 = 1.1$.

p_0 = 1.0000000000000000, $f[p_0]$ = 1.
p_1 = 1.1000000000000000, $f[p_1]$ = 0.6640000000000015
p_2 = 1.2976190476190490, $f[p_2]$ = -0.1417166072778375
p_3 = 1.2628600508339780, $f[p_3]$ = 0.007687910275651078
p_4 = 1.2646486450005470, $f[p_4]$ = 0.0000412493363857180 4
p_5 = 1.2646582934370370, $f[p_5]$ = $-1.442380259675247 \times 10^{-8}$
p_6 = 1.2646582900644130, $f[p_6]$ = $2.664535259100376 \times 10^{-14}$
p_7 = 1.2646582900644200, $f[p_7]$ = 0.

$$p_8 = 1.2646582900644200, \quad f[p_8] = 0.$$

$$p = 1.26465829006442$$
$$\Delta p = \pm 0.$$
$$f[p] = 0.$$

Mathematica can solve for the roots symbolically.

$$0 = -4 + 17x - 16x^2 + 4x^3$$

$$x \to \frac{4}{3} + \frac{13}{6\left(8 + 3 i \sqrt{237}\right)^{1/3}} + \frac{1}{6}\left(8 + 3 i \sqrt{237}\right)^{1/3}$$

$$x \to \frac{4}{3} - \frac{13\left(1 + i\sqrt{3}\right)}{12\left(8 + 3 i \sqrt{237}\right)^{1/3}} - \frac{1}{12}\left(1 - i\sqrt{3}\right)\left(8 + 3 i \sqrt{237}\right)^{1/3}$$

$$x \to \frac{4}{3} - \frac{13\left(1 - i\sqrt{3}\right)}{12\left(8 + 3 i \sqrt{237}\right)^{1/3}} - \frac{1}{12}\left(1 + i\sqrt{3}\right)\left(8 + 3 i \sqrt{237}\right)^{1/3}$$

The answers can be manipulated into real expressions.

$$x \to \frac{4}{3} + \frac{1}{3}\sqrt{13}\,\mathrm{Cos}\left[\frac{1}{3}\,\mathrm{ArcTan}\left[\frac{3\sqrt{237}}{8}\right]\right]$$

$$x \to \frac{4}{3} - \frac{1}{6}\sqrt{13}\,\mathrm{Cos}\left[\frac{1}{3}\,\mathrm{ArcTan}\left[\frac{3\sqrt{237}}{8}\right]\right] - \frac{1}{2}\sqrt{\frac{13}{3}}\,\mathrm{Sin}\left[\frac{1}{3}\,\mathrm{ArcTan}\left[\frac{3\sqrt{237}}{8}\right]\right]$$

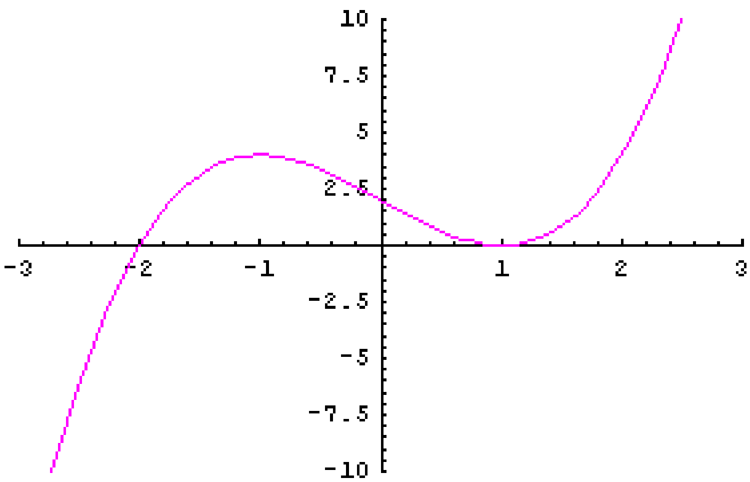
$$x \to \frac{4}{3} - \frac{1}{6}\sqrt{13}\,\mathrm{Cos}\left[\frac{1}{3}\,\mathrm{ArcTan}\left[\frac{3\sqrt{237}}{8}\right]\right] + \frac{1}{2}\sqrt{\frac{13}{3}}\,\mathrm{Sin}\left[\frac{1}{3}\,\mathrm{ArcTan}\left[\frac{3\sqrt{237}}{8}\right]\right]$$

The answers can be expressed in decimal form.

$$x \to 2.406803251324165$$
$$x \to 0.328538458611415$$
$$x \to 1.26465829006442$$

These answers are in agreement with the ones we found with the secant method.

Example 2. Use Newton's method to find the roots of the cubic polynomial $f[x] = x^3 - 3x + 2$.
2 (a) Fast Convergence. Investigate quadratic convergence at the simple root $p = -2$, using the starting value $p_0 = -2.6$ and $p_1 = -2.4$
2 (b) Slow Convergence. Investigate linear convergence at the double root $p = 1$, using the starting value $p_0 = 1.4$ and $p_1 = 1.2$
Solution 2.
Graph the function.



$$f[x] = 2 - 3x + x^3$$

The secant iteration formula $g[x_0, x_1]$ is

$$p_2 = g[p_0, p_1] = p_1 - \frac{(-p_0 + p_1)(2 - 3p_1 + p_1^3)}{3p_0 - p_0^3 - 3p_1 + p_1^3}$$

$$p_2 = g[p_0, p_1] = \frac{-2 + p_0^2 p_1 + p_0 p_1^2}{-3 + p_0^2 + p_0 p_1 + p_1^2}$$

2 (a) Fast Convergence. Investigate quadratic convergence at the simple root $p = -2$, using the starting value $p_0 = -2.6$ and $p_1 = -2.4$
First, do the iteration one step at a time.
Notice that convergence is fast and the sequence is converging to the simple root $p = -2$

$$p_0 = -2.6000000000000000, \quad f[p_0] = -7.776$$
$$p_1 = -2.4000000000000000, \quad f[p_1] = -4.623999999999999$$
$$p_2 = -2.1065989847715740, \quad f[p_2] = -1.028782245156645$$
$$p_3 = -2.0226414123070680, \quad f[p_3] = -0.2068601188187564$$
$$p_4 = -2.0015110973304850, \quad f[p_4] = -0.0136135799156829$$
$$p_5 = -2.0000225364837550, \quad f[p_5] = -0.0002028314011663923$$
$$p_6 = -2.0000000226858160, \quad f[p_6] = -2.041723501378101 \times 10^{-7}$$

$p_7 = -2.000000000003410, \quad f[p_7] = -3.065991904804832 \times 10^{-12}$
$p_8 = -2.000000000000000, \quad f[p_8] = 0.$

$p = -2.$
$\Delta p = \pm 3.40616 \times 10^{-13}$
$f[p] = 0.$

At the simple root $p = -2$, the order of convergence is known to be $\frac{\sqrt{5}+1}{2}$. We can explore the ratio $\frac{|E_{k+1}|}{(|E_k|)^{\frac{\sqrt{5}+1}{2}}}$ for k sufficiently large.

k	p_k	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{(\lvert E_k\rvert)^{\frac{\sqrt{5}+1}{2}}}$
0	-2.6	0.6	0.914152831045763
1	-2.4	0.4	0.469497763808824
2	-2.10659898477157	0.106598984771574	0.847290025710246
3	-2.02264141230707	0.0226414123070677	0.693608598604584
4	-2.00151109733049	0.00151109733048571	0.82582278936615
5	-2.00002253648376	0.0000225364837556086	0.749795034971821
6	-2.00000002268582	$2.26858163365762 \times 10^{-8}$	0.794965297614188
7	-2.00000000000034	$3.40172334745148 \times 10^{-13}$	

2 (b) Slow Convergence. Investigate linear convergence at the double root $p = 1$, using the starting value $p_0 = 1.4$ and $p_1 = 1.2$
First, do the iteration one step at a time.
Notice that convergence is slow, but the sequence is converging to the double root $p = 1$

$p_0 = 1.400000000000000, \quad f[p_0] = 0.544$
$p_1 = 1.200000000000000, \quad f[p_1] = 0.128000000000001$
$p_2 = 1.138461538461538, \quad f[p_2] = 0.06016932180245793$
$p_3 = 1.083873738458235, \quad f[p_3] = 0.02169444731993009$
$p_4 = 1.053093855030146, \quad f[p_4] = 0.00860654164364627$
$p_5 = 1.032853156857141, \quad f[p_5] = 0.003273449141453444$
$p_6 = 1.020429427842578, \quad f[p_6] = 0.001260611023022662$
$p_7 = 1.012648628383645, \quad f[p_7] = 0.0004819870261905113$
$p_8 = 1.007832125928220, \quad f[p_8] = 0.0001845070294745899$
$p_9 = 1.004844770239372, \quad f[p_9] = 0.0000705291114886375$

p_{10} = 1.0029962056563540, $f[p_{10}]$ = 0.0000269586426877577 2
p_{11} = 1.0018524311354370, $f[p_{11}]$ = 0.0000103008599539045 1
p_{12} = 1.0011451424556020, $f[p_{12}] = 3.935555414669167 \times 10^{-6}$
p_{13} = 1.0007078383523180, $f[p_{13}] = 1.503460050900074 \times 10^{-6}$
p_{14} = 1.0004375079782810, $f[p_{14}] = 5.743234381405671 \times 10^{-7}$
p_{15} = 1.0002704097628860, $f[p_{15}] = 2.193840922259938 \times 10^{-7}$
p_{16} = 1.0001671282002210, $f[p_{16}] = 8.38001743552752 \times 10^{-8}$
p_{17} = 1.0001032931001970, $f[p_{17}] = 3.200949572068623 \times 10^{-8}$
p_{18} = 1.0000638394877910, $f[p_{18}] = 1.222670054090713 \times 10^{-8}$
p_{19} = 1.0000394552943670, $f[p_{19}] = 4.670222208957853 \times 10^{-9}$
p_{20} = 1.0000243848347330, $f[p_{20}] = 1.783875047678407 \times 10^{-9}$
p_{21} = 1.0000150707032740, $f[p_{21}] = 6.813818398399007 \times 10^{-10}$
p_{22} = 1.0000093142233370, $f[p_{22}] = 2.60265142770777 \times 10^{-10}$
p_{23} = 1.0000057565133660, $f[p_{23}] = 9.94122562048005 \times 10^{-11}$
p_{24} = 1.0000035577342030, $f[p_{24}] = 3.797251402204438 \times 10^{-11}$
p_{25} = 1.0000021987901450, $f[p_{25}] = 1.450395359370305 \times 10^{-11}$
p_{26} = 1.0000013589405250, $f[p_{26}] = 5.540456982089381 \times 10^{-12}$
p_{27} = 1.0000008398182570, $f[p_{27}] = 2.115863040330623 \times 10^{-12}$
p_{28} = 1.0000005190819480, $f[p_{28}] = 8.08020317322189 \times 10^{-13}$
p_{29} = 1.0000003209224530, $f[p_{29}] = 3.090860900556436 \times 10^{-13}$
p_{30} = 1.0000001981641000, $f[p_{30}] = 1.176836406102666 \times 10^{-13}$

p = 1.0000001981641
$\Delta p = \pm 1.22758 \times 10^{-7}$
$f[p] = 1.176836406102666 \times 10^{-13}$

Real roots can also be found symbolically.

$f[x] = (-1 + x)^2 (2 + x)$

$x \to -2$

$x \to 1$

$x \to 1$

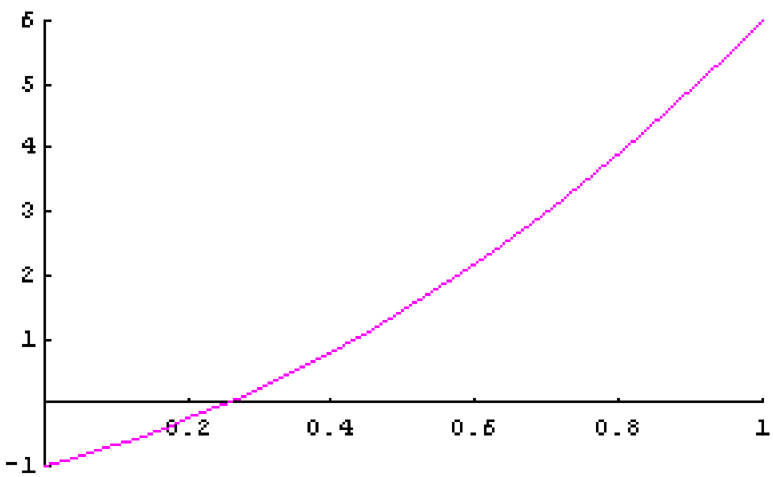
Example 3. Fast Convergence Find the solution to $3\,\mathrm{Exp}[x] - 4\,\mathrm{Cos}[x] = 0$.
Use the Secant Method and the starting approximations $p_0 = 1.0$ and $p_1 = 0.9$.
Solution 3.

$$f[x] = 3\,e^x - 4\,\mathrm{Cos}[x]$$

$$p_{k+1} = g[p_k, p_{k-1}] = p_k - \frac{f[p_k]\,(p_k - p_{k-1})}{f[p_k] - f[p_{k-1}]}$$

$p_0 = 1.0000000000000000$, $f[p_0] = 5.993636261904577$
$p_1 = 0.9000000000000000$, $f[p_1] = 4.892369460388192$
$p_2 = 0.4557507541631453$, $f[p_2] = 1.140348050580424$
$p_3 = 0.3207305327599215$, $f[p_3] = 0.3383810520017154$
$p_4 = 0.2637602526350550$, $f[p_4] = 0.0437823138992921$
$p_5 = 0.2552935134242091$, $f[p_5] = 0.00216446644720031$
$p_6 = 0.2548531741452456$, $f[p_6] = 0.0000151993311736534$
$p_7 = 0.2548500601244337$, $f[p_7] = 5.345576425952459 \times 10^{-9}$
$p_8 = 0.2548500590288531$, $f[p_8] = 1.332267629550188 \times 10^{-14}$
$p_9 = 0.2548500590288503$, $f[p_9] = 4.440892098500626 \times 10^{-16}$

$f[x] = 3\,e^x - 4\,\mathrm{Cos}[x]$
$p = 0.2548500590288503$
$\Delta p = \pm 2.72005 \times 10^{-15}$
$f[p] = 4.440892098500626 \times 10^{-16}$



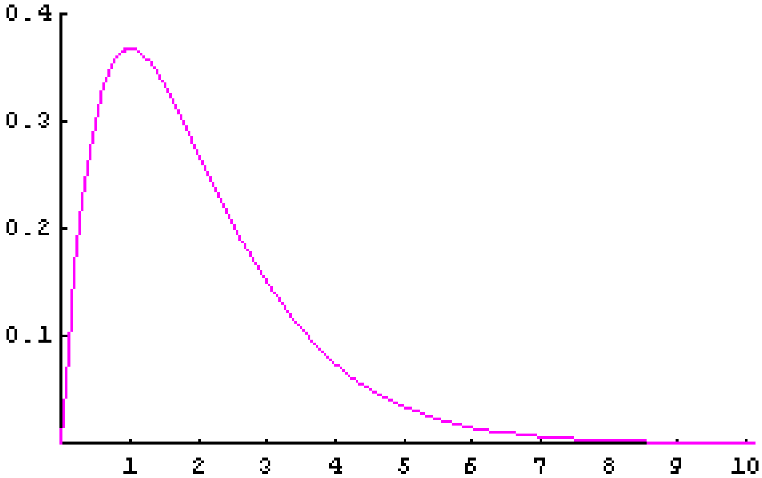
Example 4. NON Convergence, Diverging to Infinity Find the solution to $x e^{-x} = 0$.
Use the Secant Method and the starting approximations $p_0 = 1.5$ and $p_1 = 2.0$.
Solution 4.

$$f[x] = e^{-x} x$$

$$p_{k+1} = g[p_k, p_{k-1}] = p_k - \frac{f[p_k]\,(p_k - p_{k-1})}{f[p_k] - f[p_{k-1}]}$$

$p_0 = 1.5000000000000000$, $f[p_0] = 0.3346952402226448$
$p_1 = 2.0000000000000000$, $f[p_1] = 0.2706705664732254$
$p_2 = 4.1137988733263970$, $f[p_2] = 0.06724235336053466$
$p_3 = 4.8125063367527000$, $f[p_3] = 0.03911347194109026$
$p_4 = 5.7840655807283130$, $f[p_4] = 0.017792846977682$
$p_5 = 6.5948675617902740$, $f[p_5] = 0.009017610805551323$
$p_6 = 7.4280641347491580$, $f[p_6] = 0.004414773021052734$
$p_7 = 8.2272175969786900$, $f[p_7] = 0.002198962281747 66$
$p_8 = 9.0202945008851600$, $f[p_8] = 0.001090828785953443$
$p_9 = 9.8009866463165800$, $f[p_9] = 0.0005429444270753301$
$p_{10} = 10.5746397772280500$, $f[p_{10}] = 0.0002702451314765831$
$p_{11} = 11.3413303974637500$, $f[p_{11}] = 0.0001346439343778478$
$p_{12} = 12.1026086440264300$, $f[p_{12}] = 0.00006710932014891208$
$p_{13} = 12.8590927989190900$, $f[p_{13}] = 0.00003346395639037461$
$p_{14} = 13.6114981808615900$, $f[p_{14}] = 0.00001669196168180826$
$p_{15} = 14.3603132158530000$, $f[p_{15}] = 8.32835299700422 \times 10^{-6}$
$p_{16} = 15.1059717199630000$, $f[p_{16}] = 4.156315487897552 \times 10^{-6}$

$f[x] = e^{-x} x$
$p = 15.105971719963$
$\Delta p = \pm 0.745659$
$f[p] = 4.156315487897552 \times 10^{-6}$


0.4
0.3
0.2
0.1
1
2
3
4
5
6
7
8
9
10