1. Fixed Point Iteration

A fundamental principle in computer science is *iteration*. As the name suggests, a process is repeated until an answer is achieved. Iterative techniques are used to find roots of equations, solutions of linear and nonlinear systems of equations, and solutions of differential equations.

A rule or function $g(x)$ for computing successive terms is needed, together with a starting value p_0. Then a sequence of values $\{p_k\}$ is obtained using the iterative rule $p_{k+1} = g(p_k)$. The sequence has the pattern

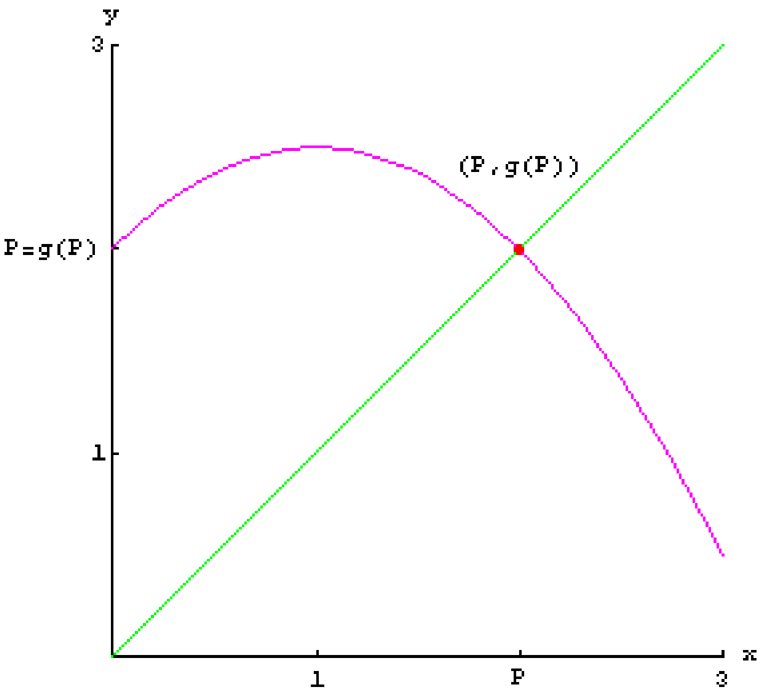
$$
\begin{aligned}
&p_0 \quad \text{(starting value)} \\
&p_1 = g(p_0) \\
&p_2 = g(p_1) \\
&\quad\vdots \\
&p_k = g(p_{k-1}) \\
&p_{k+1} = g(p_k) \\
&\quad\vdots
\end{aligned}
$$

What can we learn from an unending sequence of numbers? If the numbers tend to a limit, we suspect that it is the answer.

Definition (FixedPoint). A fixed point of a function $g(x)$ is a number P such that $P = g(P)$.

Caution. A fixed point is **not** a root of the equation $0 = g(x)$, it is a solution of the equation $x = g(x)$.

Geometrically, the fixed points of a function $g(x)$ are the point(s) of intersection of the curve $y = g(x)$ and the line $y = x$.



Theorem (Fixed Point Theorem). Assume that the following hypothesis hold true.

(a) P is a fixed point of a function g,

(b) $g, g' \in C[a, b]$,

(c) K is a positive constant,

(d) $p_0 \in (a, b)$, and

(e) $g(x) \in [a, b]$ for all $x \in [a, b]$.

Then we have the following conclusions.

(i). If the range of the mapping $y = g(x)$ satisfies $y \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.

(ii). If $|g'(x)| \le K < 1$ for all $x \in [a, b]$, then the iteration $p_n = g(p_{n-1})$ will converge to the unique fixed point $P \in (a, b)$. In this case, P is said to be an attractive fixed point.

(iii). If $|g'(x)| > 1$ for all $x \in [a, b]$, then the iteration $p_n = g(p_{n-1})$ will not converge to P.

In this case, P is said to be a repelling fixed point and the iteration exhibits local divergence.

Corollary. Assume that g satisfies hypothesis **(a)-(e)** of the previous theorem. Bounds for the error involved when using p_n to approximate P are given by

$$|P - p_n| \le K^n |P - p_0| \quad \text{for } n \ge 1,$$

and

$$\left|P - p_n\right| \le \frac{K^n}{1 - K} \left|p_1 - p_0\right| \quad \text{for } n \ge 1.$$

Graphical Interpretation of Fixed-point Iteration

Since we seek a fixed point P to $g(x)$, it is necessary that the graph of the curve $y = g(x)$ and the line $y = x$ intersect at the point (P, P).
The following animations illustrate two types iteration: monotone and oscillating.

Algorithm (Fixed Point Iteration). To find a solution to the equation $x = g(x)$ by starting with p_0 and iterating $p_n = g(p_{n-1})$.

Example 1. Use fixed point iteration to find the fixed point(s) for the function $g(x) = 1 + x - \frac{x^2}{3}$.

Solution 1.

Example 2. Convergence: Monotone Decreasing Find the solution to $x = g[x] = \frac{x^2}{4} + \frac{x}{2}$. Use the starting approximation $p_0 = 1.95$

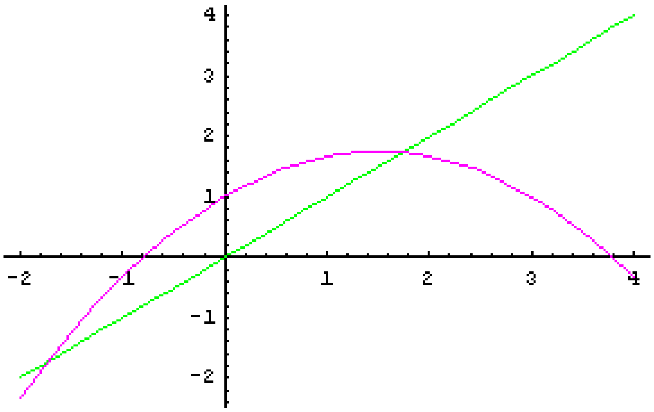
Solution 2.

Example 3. Divergence: Spiral Find the solution to $x = g[x] = -\frac{x^2}{4} - \frac{x}{2} + 4$. Use the starting approximation $p_0 = 1.97$

Solution 3.

Example 1. Use fixed point iteration to find the fixed point(s) for the function $g(x) = 1 + x - \frac{x^2}{3}$.

Solution 1.
Plot the function and determine graphically that there are two solutions to the equation $x = g(x)$.



The function is $g[x] = 1 + x - \frac{x^2}{3}$

Use fixed point iteration to find a numerical approximation.
First, do the iteration one step at a time. Type each of the following commands in a separate cell and execute them one at a time.

$p_0 = 3.0$
3.

$p_1 = g[p_0]$
1.

$p_2 = g[p_1]$
1.66667

$p_3 = g[p_2]$
1.74074

$p_4 = g[p_3]$
1.73068

$p_5 = g[p_4]$
1.73226

Remark. The distinguishing property for determining convergence is the size of $|g'[p]|$.
If p_0 is near the fixed point p and $|g'[p]| < 1$ then the iteration will converge to p.
If p_0 is near the fixed point p and $|g'[p]| > 1$ then the iteration will not converge to p.

The function is $g[x] = 1 + x - \frac{x^2}{3}$

The derivative is $g'[x] = 1 - \frac{2x}{3}$

$$g'[\sqrt{3}] = 1 - \frac{2}{\sqrt{3}}$$
$$|g'[\sqrt{3}]| = -1 + \frac{2}{\sqrt{3}}$$
$$|g'[\sqrt{3}]| = 0.154701$$

$$g'[-\sqrt{3}] = 1 + \frac{2}{\sqrt{3}}$$
$$|g'[-\sqrt{3}]| = 1 + \frac{2}{\sqrt{3}}$$
$$|g'[-\sqrt{3}]| = 2.1547$$

If p_0 is near the fixed point $p = \sqrt{3}$ and $|g'[x]| < 1$ in the neighborhood, then the iteration will
converge to $p = \sqrt{3}$.
If p_0 is near the fixed point $p = -\sqrt{3}$ and $|g'[x]| > 1$ in the neighborhood, then the iteration
will not converge to $p = -\sqrt{3}$.

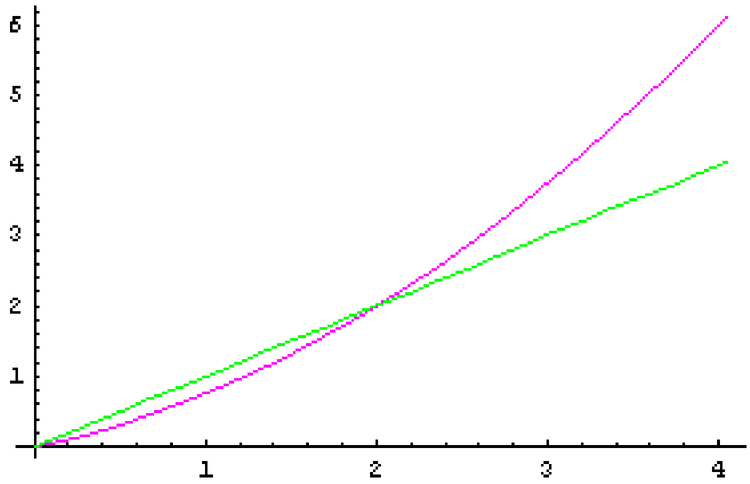
Example 2. Convergence: Monotone Decreasing Find the solution to $x = g[x] = \frac{x^2}{4} + \frac{x}{2}$. Use the starting approximation $p_0 = 1.95$

Solution 2.

$p_0 = 1.950000000000000$
$p_1 = 1.925625000000000$
$p_2 = 1.889820410156250$
$p_3 = 1.837765500738910$
$p_4 = 1.763228259295990$
$p_5 = 1.658857603242980$
$p_6 = 1.517380938580760$
$p_7 = 1.334301697482430$
$p_8 = 1.112241103717340$
$p_9 = 0.865390620058263$
$p_{10} = 0.619920541350338$
$p_{11} = 0.406035640072193$
$p_{12} = 0.244234055288306$
$p_{13} = 0.137029596084796$
$p_{14} = 0.073209075593188$
$p_{15} = 0.037944429983896$
$p_{16} = 0.019332159933649$
$p_{17} = 0.009759513068750$
$p_{18} = 0.004903568558210$
$p_{19} = 0.002457795525256$
$p_{20} = 0.001230407952339$
$p_{21} = 0.000615582452102$
$p_{22} = 0.000307885961490$
$p_{23} = 0.000153966679186$
$p_{24} = 0.000076989266028$
$p_{25} = 0.000038496114851$
$p_{26} = 0.000019248427913$
$p_{27} = 9.624306582002880 \times 10^{-6}$
$p_{28} = 4.812176447820730 \times 10^{-6}$
$p_{29} = 2.406094013170910 \times 10^{-6}$
$p_{30} = 1.203048453907550 \times 10^{-6}$

The function is $g[x] = \frac{x}{2} + \frac{x^2}{4}$

$p = 1.203048453907550 \times 10^{-6}$
$g[p] = 6.015245887851730 \times 10^{-7}$



$$g[0] = 0$$

$$g'[0] = \frac{1}{2}$$

Since $|g'(x)| < 1$ for all $x \in (-3, 1)$, the iteration $p_n = g(p_{n-1})$ will converge to the fixed point $p = 0$. In this case, $p = 0$ is said to be an attractive fixed point.

$$g'[x] = \frac{1}{2} + \frac{x}{2}$$

Solve $-1 < g'[x] < 1$

Get $-3 < x < 1$

$$g[2] = 2$$

$$g'[2] = \frac{3}{2}$$

Since $|g'(x)| > 1$ for all $x \in (1, \infty)$, then the iteration $p_n = g(p_{n-1})$ will not converge to $p = 2$. In this case, $p = 2$ is said to be a repelling fixed point and the iteration exhibits local divergence.

$$g'[x] = \frac{1}{2} + \frac{x}{2}$$

Solve $g'[x] > 1$

Get $x > 1$

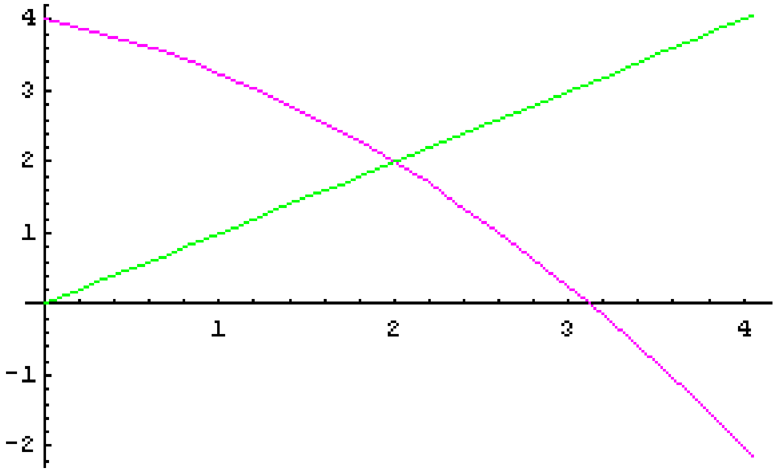
Example 3. Divergence: Spiral Find the solution to $x = g[x] = -\frac{x^2}{4} - \frac{x}{2} + 4$. Use the starting approximation $p_0 = 1.97$

Solution 3.

$$
\begin{aligned}
p_0 &= 1.970000000000000\\
p_1 &= 2.044775000000000\\
p_2 &= 1.932336299843750\\
p_3 &= 2.100350956154670\\
p_4 &= 1.846955987167710\\
p_5 &= 2.223710401782480\\
p_6 &= 1.651922811359860\\
p_7 &= 2.491826350647300\\
p_8 &= 1.201787184231280\\
p_9 &= 3.038033298838720\\
p_{10} &= 0.173571769367422\\
p_{11} &= 3.905682325535960\\
p_{12} &= -1.766429769768960\\
p_{13} &= 4.103146352002970\\
p_{14} &= -2.260525672490310\\
p_{15} &= 3.852768757248210\\
p_{16} &= -1.637341152831090\\
p_{17} &= 4.148449063726990\\
p_{18} &= -2.376631940447820\\
p_{19} &= 3.776221125134720\\
p_{20} &= -1.453072059045790
\end{aligned}
$$

The function is $g[x] = 4 - \frac{x}{2} - \frac{x^2}{4}$

$$
\begin{aligned}
p &= -1.453072059045790\\
g[p] &= 4.198681427328000
\end{aligned}
$$



$$g[2] = 2$$

$$g'[2] = -\frac{3}{2}$$

Since $|g'(x)| > 1$ for all $x \in (1, \infty)$, then the iteration $p_n = g(p_{n-1})$ will not converge to $p = 2$. In this case, $p = 2$ is said to be a repelling fixed point and the iteration exhibits local divergence.

$$g'[x] = -\frac{1}{2} - \frac{x}{2}$$

Solve $g'[x] < -1$

Get $x > 1$